

Noncommutative Solitons on Orbifolds

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In the noncommutative field theory of open strings in a B -field, D-branes arise as solitons described as projection operators or partial isometries in a C^* algebra. We discuss how D-branes on orbifolds fit naturally into this algebraic framework, through the examples of \mathbb{R}^n/G , $\mathbf{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and \mathbf{T}^n/G . We also propose a framework for formulating D-branes on asymmetric orbifolds.

1. Introduction

Field theory on a noncommutative space is proving to be a useful limiting case of string theory [1,2], which still preserves some interesting aspects of stringy structure, for instance UV/IR mixing [3,4] and T-duality [1,5,6,7,8,9,2]. In some respects, the scale set by the noncommutativity plays a role similar to the string scale – as both a regulator and a source of nonlocality in the theory. In this setting, the role of the algebra of functions on a space is played by a noncommutative C^* algebra \mathcal{A} . The simplest example is the noncommutative plane \mathbb{R}^{2d} .

The algebraic structure of noncommutative geometry allows a particularly simple description of D-branes as noncommutative solitons [10,11,12], and clarifies [13,14,15] the relation between D-branes and K-theory [16,17,18,19]. As we review in section 2, D-branes of even codimension on \mathbb{R}^d are solitons in the effective field theory of open strings. The soliton configurations are described by projection operators or partial isometries in the C^* algebra \mathcal{A} . One of the goals of this work is to generalize the construction (and classification) of D-branes as solitons to the next simplest string target spaces, namely tori and orbifolds.

The study of D-branes on orbifolds was begun in [20,21,22]. For noncompact orbifolds \mathbb{R}^d/G , the spectrum of branes and the field theories on them are determined by the representation theory of G [21,22,23,24]. Placing an image D-brane on each leaf of the covering space \mathbb{R}^d yields a set of branes transforming in the regular representation of G . When taken to the orbifold point, this brane splits up into a set of *fractional branes* according to the decomposition of the regular representation into irreducible representations; these fractional branes are the basic objects in the classification of branes on the orbifold. These basic branes can again be thought of in terms of a projection of branes on the covering space \mathbb{R}^d .

The appropriate and universal definition of the quotient of an algebra \mathcal{A} by the action of a group G is actually to consider an enlarged algebra known as the *crossed product* $\mathcal{A} \rtimes G$ [21,25,26], which we define in section 3. Multiplication of elements in the crossed product involves the multiplication laws in both G and \mathcal{A} , as well as the action of G on \mathcal{A} . Physical string states belong to the G -invariant subalgebra of the crossed product.

It is thus a straightforward consequence of this observation, combined with the construction of [12], that given an algebraic description of D-brane solitons on a space \mathcal{Y} as operators in an algebra \mathcal{A} , then the description of D-brane solitons on \mathcal{Y}/G will arise upon taking the crossed product $\mathcal{A} \rtimes G$, and constructing appropriate projectors or partial

isometries there. We illustrate this procedure in sections 3 and 4 by constructing projectors and partial isometries for the orbifolds \mathbb{R}^d/G , $\mathbf{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, and \mathbf{T}^d/G . Tachyon condensation on the two-dimensional noncommutative torus has also been discussed recently in [27,28,29].

In a companion paper [30], we intend to apply the tools of noncommutative geometry to relate the classification of D-brane charges to the K-theory and the cohomology of the corresponding noncommutative algebras. The application of noncommutative differential geometry [31] (for reviews, see [32,33,34,35]) in the context of D-branes on noncommutative spaces has been developed in the works [1,5,6,7,36,26] and we will build on this work. We will show how the Chern character of noncommutative differential geometry, as defined by A. Connes [31,32] is related to the Chern-Simons couplings of branes. We will also exhibit a set of cyclic cocycles for certain crossed product algebras that measure the topological invariants of branes on orbifolds, and are thus suitable for use in constructing their Chern-Simons couplings.

Related work by R. Gopakumar and M. Headrick has been announced in [37].

2. Summary of D-branes as noncommutative solitons

2.1. Noncommutative geometry

Suppose we have a closed string background $\mathcal{X} \times \mathbb{R}^{2d}$. Consider some number of $D9$ -branes in type IIA string theory, or $D9-\overline{D9}$ pairs in type IIB string theory (the extension to the bosonic string is obvious). The effect of turning on a nondegenerate B field along \mathbb{R}^{2d} can be absorbed [1,38,39,2] into a description of the dynamics in terms of noncommutative geometry. For other expository discussions of the application of noncommutative geometry in this context, see [25,40,41].

The multiplication $*$ of fields depending on noncommutative \mathbb{R}^{2d} is induced by the algebra of coordinates

$$[x^{2i-1}, x^{2i}] := x^{2i-1} * x^{2i} - x^{2i} * x^{2i-1} = -i\theta_i, \quad (2.1)$$

where the θ^i , $i = 1, \dots, d$ are the skew eigenvalues of the parameter Θ^{ij} appearing in the Moyal product. Then we have the Moyal algebra of functions \mathcal{A} , for which we can map any function f to a corresponding operator using Weyl ordering

$$f \rightarrow U(f) := \int d^{2d}p \tilde{f}(p) e^{ip \cdot \hat{x}} \quad (2.2)$$

where \tilde{f} is the commutative Fourier transform and \hat{x} is the quantization of x using the standard Heisenberg representation. If we restrict to functions of rapid decrease the set of operators $U(f)$ is a C^* -algebra isomorphic to the algebra of compact operators \mathcal{K} (for any d, Θ) [42].

It is also convenient to introduce the Hilbert space \mathcal{H} on which \mathcal{A} acts, and its standard number basis $|\vec{n}\rangle$ for the raising and lowering operators $a_i^\dagger = (x^{2i} - ix^{2i-1})/\sqrt{2\theta_i}$, $a_i = (x^{2i} + ix^{2i-1})/\sqrt{2\theta_i}$. Translations on \mathbb{R}^{2d} are generated by the unitary operators

$$\begin{aligned} \mathcal{T}(z) &= \exp[z^i a_i^\dagger - \bar{z}^i a_i] \\ \mathcal{T}(z)\mathcal{T}(w) &= e^{\frac{1}{2}(z\bar{w} - w\bar{z})}\mathcal{T}(z+w) ; \end{aligned} \quad (2.3)$$

in particular one has $\mathcal{T}(z)a\mathcal{T}^{-1}(z) = a - z$. Associated to the translation operators are the coherent states

$$|z\rangle = \mathcal{T}(z)|0\rangle . \quad (2.4)$$

In sections 3 and 4, we will generalize the discussion to other noncommutative manifolds besides \mathbb{R}^{2d} : noncompact orbifolds, noncommutative tori and toroidal orbifolds.

2.2. Effective D-brane dynamics

Integrating out massive string modes results in a low energy effective action that is a noncommutative field theory of the open string tachyon and gauge field [1,39,2]; on the type IIA D9-brane, we have

$$\begin{aligned} S = \frac{c}{G_s} \int_{\mathcal{X}} d^{10-2d}x \sqrt{G} \text{Tr} \Big[\frac{1}{2} f(T^2 - 1) G^{\mu\nu} D_\mu T D_\nu T - V(T^2 - 1) \\ - \frac{1}{4} h(T^2 - 1) (F_{\mu\nu} + \Phi_{\mu\nu})(F^{\mu\nu} + \Phi^{\mu\nu}) + \dots \Big] \end{aligned} \quad (2.5)$$

where G_s and $G_{\mu\nu}$ are the coupling and metric felt by open strings, *c.f.* [43,44,2]; Tr is the trace of the operator on Hilbert space; and x^μ runs over both the commuting coordinate directions x^a and noncommuting coordinate directions x^i on \mathcal{X} . The antisymmetric tensor Φ incorporates the possibility of noncommutativity of derivatives [45]

$$[\partial_i, \partial_j] = i\Phi_{ij} . \quad (2.6)$$

The functions f, h, V are unknown but satisfy certain crucial properties in accord with the conjectures of [46,47]. The Dirac-Born-Infeld extension of (2.5) has been considered for tachyon dynamics in [2,47]; and for noncommutative gauge theory in [2,9,8,48]. Similarly,

on the type IIB $D9-\overline{D9}$ system, we have a pair of gauge fields A^\pm acting on a complex tachyon field T as

$$\begin{aligned} D_\mu T &= \partial_\mu T + i(A_\mu^+ T - T A_\mu^-) \\ \overline{D}_\mu \bar{T} &= \partial_\mu \bar{T} - i(\bar{T} A_\mu^+ - A_\mu^- \bar{T}) , \end{aligned} \quad (2.7)$$

the tachyon potential is $V(T, \bar{T}) = U(\bar{T}T - 1) + U(T\bar{T} - 1)$, and the gauge kinetic term is suitably generalized [49]. Here and henceforth $\overline{}$ denotes the C^* algebra adjoint. It is convenient to introduce the gauge field (in complex coordinates) in the noncommutative directions

$$C_j = \theta_j^{-\frac{1}{2}} a_j^\dagger + iA_j \quad , \quad \bar{C}_j = \theta_j^{-\frac{1}{2}} a_j - i\bar{A}_j \quad , \quad (2.8)$$

in terms of which $(F + \Phi)_{2j-1, 2j} = [C_j, \bar{C}_j]$ (note that this implies $\Phi = -\Theta^{-1}$ for F a compact operator [45]). The global minimum of the action

$$|T| = 1 \quad , \quad C_j = \theta_j^{-\frac{1}{2}} a_j^\dagger \quad (2.9)$$

is to be identified with the closed string vacuum, having no perturbative open string excitations [46].

In the limit $\alpha' B_{ij} \rightarrow \infty$ (or equivalently, $\Theta^{ij}/\alpha' \rightarrow 0$), by rescaling the coordinates to remove θ_i from the star product one sees that the action reduces to the nonderivative terms. It turns out that there are interesting soliton solutions of the equations of motion. In the IIA case, one finds solutions in terms of a projection operator [10]

$$T = \mathbf{1} - \mathbf{P} \quad , \quad \mathbf{P}^2 = \mathbf{P} . \quad (2.10)$$

A rank k projection operator gives $U(k)$ gauge symmetry on the lower-dimensional unstable $D(9 - 2d)$ brane. The dynamical degrees of freedom on this lower-dimensional brane are operators mapping the kernel of T to itself.

For type IIB, the complex tachyon T must satisfy [50] the defining equation of a *partial isometry*

$$T\bar{T}T = T . \quad (2.11)$$

The net brane charge is the index of T ; we will assume for simplicity that T has vanishing cokernel. The dynamical degrees of freedom on the lower dimensional brane again arise from operators mapping the kernel of T to itself.

In both IIA and IIB situations, the tension and effective actions of these soliton solutions turn out to be exactly those of lower-dimensional D-branes [12]. One might be concerned that this remarkable result, seemingly at leading order in the expansion in inverse powers of $\alpha' B$, receives corrections at each order that spoil the precise agreement. However, recently it has been shown [49] that the structure of the effective action (2.5) is such that a suitable gauge field can be found for which (2.10) and (2.11) are *exact*.¹ The idea is that, starting from the closed string vacuum solution (2.9), one may use partial isometries U to construct new solutions to the equations of motion

$$T \rightarrow UT\bar{U} = U\bar{U} \quad , \quad C \rightarrow UC\bar{U} = U\theta^{-\frac{1}{2}}a^\dagger\bar{U}. \quad (2.12)$$

These are almost gauge transformations, in that $U\bar{U} = \mathbf{1} - \mathbf{P}_n$, $\bar{U}U = \mathbf{1} - \mathbf{P}_m$ shows that the transformation is unitary in the orthogonal complement to a (finite-dimensional) kernel and cokernel; hence U is ‘almost’ a unitary transformation. In this sense, the transformation is a bit like that which generates a zero-size instanton, or a vertex operator in WZW/Chern-Simons theory [52]– namely, pure gauge outside a small ‘singular’ region which carries topological winding (hence quantized action); noncommutativity even smooths the singularity. The symmetry properties allow one to show that the transformation (2.12) generates *exact* D-brane solutions of the equations of motion for *any* value of $\alpha' B$, with the correct value of the tension [49].

BPS D -branes arise in the brane-antibrane system from the noncommutative version [14] of the ABS construction [53,17,18]. Let

$$\gamma_i = \begin{pmatrix} 0 & \Gamma_i \\ \bar{\Gamma}_i & 0 \end{pmatrix} \quad (2.13)$$

be a representation of the $2d$ -dimensional Clifford algebra. Then

$$T = \frac{1}{\sqrt{\Gamma_i x^i \bar{\Gamma}_i x^i}} \Gamma_i x^i \quad (2.14)$$

satisfies the partial isometry equation $T\bar{T}T = T$, has no cokernel and is of index 1. The analysis of [14] (see also [15]) shows that this tachyon field carries the K-theory charge of a BPS $D(9 - 2d)$ brane.

In the framework of noncommutative geometry, the quantization of the point particle at the endpoint of the open string gives a Hilbert space \mathcal{H} . Open strings with the same

¹ Very similar constructions also appeared in [51].

boundary conditions are elements of $\mathcal{H} \otimes \mathcal{H}^\vee$, *i.e.* a pair of string endpoints with opposite orientations because we are allowed to ignore the string oscillators in the large $\alpha' B_{ij}$ limit. On the other hand $\mathcal{H} \otimes \mathcal{H}^\vee$ is an algebra, corresponding to the joining of string endpoints.² Upon tachyon condensation, \mathcal{H} has an orthogonal decomposition into $\ker(T)$, representing the endpoints of strings on lower-dimensional branes, and $\ker(T)^\perp$, representing string endpoints “off the branes”. Elements of the algebra \mathcal{A} split into [50]

$$\begin{pmatrix} \mathcal{U} & \mathcal{V} \\ \tilde{\mathcal{V}} & \mathcal{W} \end{pmatrix}, \quad (2.15)$$

with $\mathcal{U} \in \text{Aut}(\ker(T))$, $\mathcal{W} \in \text{Aut}(\ker(T)^\perp)$, $\mathcal{V} \in \text{Hom}(\ker(T)^\perp, \ker(T))$, and $\tilde{\mathcal{V}} \in \text{Hom}(\ker(T), \ker(T)^\perp)$. Now \mathcal{W} is the image under (2.2) of a function supported outside the location of the branes, where the tachyon has condensed to the closed string vacuum. We thus associate $\ker(T)^\perp$ with string endpoint configurations in the closed string vacuum. There are many operators in \mathcal{A} of the form \mathcal{V} , $\tilde{\mathcal{V}}$, \mathcal{W} (typically most of \mathcal{A} , in fact), that do not act only within the subspace $\ker(T)$ of \mathcal{H} corresponding to the lower-dimensional branes. Following this line of reasoning (adopted from [50]), these operators correspond to open strings with one or both endpoints trying to end in the closed string vacuum. It was shown in [12] that the related fields in the effective action have string scale mass after tachyon condensation, and so should not be considered in the effective action approximation.³ Physically, the absence of such fields is due to the different ways Gauss’ law can be satisfied at $T = 0$ (on the unstable D-brane) and $T = 1$ (in the closed string vacuum). In the presence of a D -brane, a string can end on the submanifold the D -brane occupies, and the flux sourced by the string endpoint is carried at little energy cost on the D -brane worldvolume. In the closed string vacuum, there are no light excitations capable of supporting this flux (they cost energy scaling as g_s^{-1}), and an open string endpoint must find another open string to bind to. We will henceforth make the standard assumption that stringy dynamics eliminates the unwanted open string excitations from the spectrum [46].

There are clearly many ways to condense open string tachyons to obtain k lower-dimensional D -branes; one can start with some number M of $D9$ -branes or $D9\text{-}\overline{D9}$ pairs, and distribute k tachyonic solitons in various ways among them. Although the starting

² Technically, $\mathcal{H} \otimes \mathcal{H}^\vee$ is the algebra of Hilbert-Schmidt operators on Hilbert space. The norm completion gives the C^* algebra of compact operators.

³ In the level truncation approximation of the full string field theory, [54] showed that there are no on-shell modes of the form \mathcal{W} in the linearized approximation.

point is not unique, the dynamics on the lower-dimensional branes (and in the closed string vacuum) is unique, and thus there is a sort of “universality” in the flows in configuration space resulting from tachyon condensation. Descriptions beginning with any number of unstable branes must therefore be equivalent.⁴ This is reflected in the fact that, while the algebras \mathcal{A} and $\mathcal{A} \otimes \text{Mat}_N(\mathbb{C})$ are not isomorphic, they are *Morita equivalent* – their representation theories are isomorphic. Hence their K-theories are the same, and the classification of lower-dimensional brane charges is unaffected by the freedom in the description. We will often find it useful to take advantage of this freedom – to add arbitrary Chan-Paton factors without affecting the effective dynamics – in what follows.

As pointed out in [12,50], the above discussion carries over more or less unchanged to the full string theory, at least formally. The algebra of open string fields \mathcal{A}_{str} factorizes [50] as $\mathcal{A}_{\text{str}} \rightarrow \mathcal{A}_0 \times \mathcal{A}_1$ in the limit under consideration, where \mathcal{A}_0 is the algebra of vertex operators with zero momentum in the noncommutative directions, and \mathcal{A}_1 is the algebra \mathcal{A} of noncommutative functions considered above.

2.3. The moduli space of separated branes

We would like to understand the description of the moduli space in the noncommutative framework. The fact that the noncommutative framework reproduces the correct low-energy dynamics guarantees that the moduli space will be the same as the commutative description; one can simply give an expectation value to the transverse scalars A_i to move a regular representation brane away from the origin.

Consider for simplicity noncommutative \mathbb{R}^2 . A partial isometry for N branes at the origin is

$$T_{(N)} = \frac{1}{\sqrt{a^N a^{\dagger N}}} \cdot a^N \quad . \quad (2.16)$$

Note that $T_{(N)}$ has kernel spanned by $|i\rangle$, $i = 0, \dots, N-1$, vanishing cokernel, and obeys $T_{(N)} \bar{T}_{(N)} = \mathbf{1}$, and therefore $\bar{T}_{(N)} T_{(N)} = \mathbf{1} - \mathbf{P}_N$ (where $\mathbf{P}_N = \sum_{i=0}^{N-1} |i\rangle\langle i|$ is the standard level N projector).

Moving the branes away from the origin can be achieved by giving an expectation value $A_z = \sqrt{\theta} \sum_i z_i |i\rangle\langle i|$ to the Goldstone mode A_z . A suitable family of partial isometries

⁴ The effective fields describing the $D9$ -branes parametrize the collective modes of a particular unstable field configuration; as the tachyon condenses, they lose their preferred status and merge with the totality of string field excitations, *c.f.* [46].

allows the construction of exact solutions (2.12) [49]. From (2.3)-(2.4), one sees that the partial isometry

$$T_{\{z\}} = \frac{1}{\sqrt{\mathcal{K}\bar{\mathcal{K}}}} \cdot \mathcal{K} \quad (2.17)$$

$$\mathcal{K} = \prod_{k=0}^{N-1} (a - z_k)$$

has as kernel the space spanned by the coherent states $|z_i\rangle$, $i = 0, \dots, N-1$, vanishing cokernel, and obeys $T_{\{z\}}\bar{T}_{\{z\}} = \mathbf{1}$, and also $\bar{T}_{\{z\}}T_{\{z\}} = \mathbf{1} - \mathbf{P}_{\{z\}}$ (where $\mathbf{P}_{\{z\}}$ the projector onto the linear span of the $|z_i\rangle$). This then is an appropriate partial isometry to describe the IIB case; $\bar{T}_{\{z\}}T_{\{z\}}$ is the appropriate projector for IIA branes. For large separation, the associated gauge field is approximately

$$A_z \sim \sum_{i=1}^N z_i |z_i\rangle \langle z_i| \quad (2.18)$$

in the space spanned by the $|z_i\rangle$, up to exponentially small corrections due to the fact that the coherent states are not quite orthogonal, $|\langle z_j | z_i \rangle|^2 = \exp[-|z_i - z_j|^2]$. Making a small fluctuation expansion about this background, one sees that the kinetic terms $|[A_z, \delta T]|^2$ and $|[A_z, \delta A_a]|^2$ give the right masses to the tachyon and gauge field excitations on the lower-dimensional branes. Note also that the moduli space of partial isometries (2.17), and thus the moduli space of N D-branes on \mathbb{R}^2 , is manifestly $(\mathbb{R}^2)^N / S_N$.

Above two dimensions, the description is more complicated due to the fact that the operators $\Gamma \cdot (x - x_i)$ do not commute with one another, due to both the matrix structure and the noncommutativity of the x 's. Nevertheless, it is straightforward to show that the N^{th} power of (2.14) has N dimensional kernel, and describes N branes at the origin. Furthermore, for x_i well separated relative to the scale of the noncommutativity, the product

$$T = \prod_{i=1}^N \frac{1}{\sqrt{\Gamma \cdot (x - x_i) \bar{\Gamma} \cdot (x - x_i)}} \Gamma \cdot (x - x_i) \quad (2.19)$$

is well approximated by the corresponding expression for commuting x . The latter has unit winding around each x_i (in the vicinity of each x_i , one has the ABS construction left- and right-multiplied by matrices with no winding around x_i); correspondingly, one sees that the operator (2.19) has a kernel consisting of states whose expectation value for x is concentrated around the x_i . Thus it is an appropriate partial isometry for well-separated branes.

One puzzling feature of (2.19) is that the partial isometry depends on the choice of ordering of the factors. The space of partial isometries is thus $(\mathbb{R}^{2d})^N$, and one might wonder whether it is correct to quotient by the symmetric group. However, different choices do not affect the low energy lagrangians on the lower-dimensional branes. Using the exact construction of [49], equation (2.12), one can show that the low-energy effective action on N coincident branes is the correct supersymmetric $U(N)$ gauge theory, and the symmetric group quotient is part of the gauge symmetry. This indicates that changes in the ordering of factors in (2.19) only affect the elements $\mathcal{V}, \tilde{\mathcal{V}}, \mathcal{W}$ in (2.15), and are not a matter of concern; nevertheless, it would be nice to understand better the detailed structure of the higher dimensional moduli space.

3. Noncompact (\mathbb{R}^{2d}/G) orbifolds

The algebraic structure of the noncommutative description of D-branes allows a natural incorporation of the algebraic structure of D-branes on orbifolds [21,23,24]. The latter consists of building representations of the orbifold group via an appropriate projection of the open strings for a collection of D-branes on the covering space. For noncompact orbifolds, the covering space is the noncommutative plane, whose associated algebra of operators (and its Hilbert space representation) carries a natural action of the discrete subgroup of rotations by which we wish to orbifold. In particular, we can build operators (2.10), (2.11) for a collection of D-branes; we will see that these intertwine naturally with the orbifold action to provide projectors for an arbitrary collection of (fractional) D-branes on the orbifold space.

Suppose we have a point group G which is a finite subgroup of $O(\mathbb{R}^{2d})$. To define D-branes on the orbifold \mathbb{R}^{2d}/G one must find:

- a.) A homomorphism $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$;
 - b.) A representation $\pi_{\mathcal{A}}$ of \mathcal{A} on a Hilbert space \mathcal{H} ;
 - c.) A unitary representation π_G of the group on Hilbert space \mathcal{H} ;
- such that we have a “covariant representation”:

$$\pi_G(\gamma)\pi_{\mathcal{A}}(a)\pi_G(\gamma)^{-1} = \pi_{\mathcal{A}}(\alpha_{\gamma}(a)) \quad (3.1)$$

for all $a \in \mathcal{A}$. This is simply a generalization of the construction of [21]. There, a scalar field like $T(x)$ was considered as an operator on a finite dimensional Chan-Paton space, and one imposed $\pi_G(\gamma)T(x)\pi_G(\gamma)^* = (\alpha_{\gamma}T)(x) = T(\gamma^{-1}x)$. The above generalizes to

$T(x)$ which live in a general C^* algebra, rather than the algebra of $N \times N$ matrices. In particular, there is an obvious action on functions $(\gamma \cdot f)(x) = f(\gamma^{-1}x)$ which defines the automorphism α_γ on the algebra \mathcal{A} :

$$\alpha_\gamma(U(f)) := U(\gamma \cdot f) ; \quad (3.2)$$

In particular, the operators a_i, a_i^\dagger transform as the coordinates of \mathbb{R}^{2d} under the action of G . Mathematically, a covariant representation as above defines a C^* representation of the *crossed product* algebra $\mathcal{A} \rtimes G$.

Discrete torsion is determined by an element of group cohomology; as suggested in [21] and investigated in detail in [55], it can be incorporated if we simply replace (a) by a projective unitary representation determined by a two-cocycle in $H^2(G, U(1))$. The generalization of (a)-(c) is called a *twisted* cross product. The multiplication rule is:

$$\left(\sum_\gamma a_\gamma \gamma \right) \cdot \left(\sum_\gamma b_\gamma \gamma \right) = \sum_{\gamma, \gamma'} a_\gamma \alpha_\gamma(b_{\gamma'}) \sigma(\gamma, \gamma') \gamma \gamma' \quad (3.3)$$

where σ is the $U(1)$ -valued group cocycle.

Note that physical open strings are invariant under the action of G ; for example, in an annulus diagram the sum over twisted boundary conditions in the closed string channel amounts to a projection onto G -invariant states in the open string channel. In the language of noncommutative geometry, physical string modes are operators belonging to the commutant of G in the crossed product algebra $\mathcal{A} \rtimes G$. With regard to topology, there is no difference in the K-theory of the crossed product algebra and that of its G -invariant subalgebra [56].

Let us close this general overview with a remark. In the early days of conformal field theory (*c.f.* [57]) it was noted that for some abelian orbifolds one could “orbifold twice” and recover the original theory. We can explain this phenomenon easily in the present context as a result of “Takai duality” ([58], Theorem 10.1.2). If G is a locally compact abelian group, then $\mathcal{A} \rtimes G$ has a \hat{G} action, where \hat{G} is the dual group. We can therefore gauge the \hat{G} symmetry, and in this sense we can “orbifold twice.” The Takai duality theorem says that

$$(\mathcal{A} \rtimes G) \rtimes \hat{G} = \mathcal{A} \otimes \mathcal{K}(L^2(G)) , \quad (3.4)$$

where \mathcal{K} is the algebra of compact operators. Thus, “orbifolding twice” gives back the original theory, up to Morita equivalence. The essential physical phenomenon is that the twisted sectors from \hat{G} restore the states which were projected out by G .

3.1. Type IIA

Let us consider first the type IIA $D9$ -brane. Suppose we have a covariant representation as above; since \mathcal{H} is a representation of a locally compact group G (actually, we are taking it to be finite), we can decompose \mathcal{H} into its *isotypical components*:

$$\mathcal{H} = \oplus_{\rho \in \widehat{G}} \mathcal{H}_\rho , \quad (3.5)$$

where \widehat{G} is the set of unitary irreps of G . Isotypical means the Hilbert space is a direct sum of copies of the irrep ρ .

Again one expects the tachyon field to be a projection operator in the large Θ limit, and the solutions to take the form

$$T = \mathbf{1} - \sum_{\rho \in \widehat{G}} \mathbf{P}_\rho , \quad (3.6)$$

where \mathbf{P}_ρ is a finite rank projection operator acting on the fractional brane subspace \mathcal{H}_ρ . If we think of $\mathcal{H}_\rho = \mathcal{H}_0 \otimes \rho$ where \mathcal{H}_0 is some “standard Hilbert space” (like the oscillator space, or more mathematically $\ell^2(\mathbb{N})$), and \mathbf{P}_ρ is of the form $\tilde{\mathbf{P}}_\rho \otimes 1$ with $\tilde{\mathbf{P}}_\rho$ of rank n_ρ , then the energy is proportional to

$$\sum_{\rho \in \widehat{G}} n_\rho \dim(\rho) . \quad (3.7)$$

This solution corresponds to having a collection of n_ρ fractional branes of type ρ .

More concretely, consider the $d = 1$ example of \mathbb{C}/\mathbb{Z}_N . According to (3.2), the creation operator transforms as

$$a \rightarrow \omega a \quad (3.8)$$

(with $\omega = \exp[2\pi i/N]$) under the generator of \mathbb{Z}_N . Therefore, the number basis states of Hilbert space transform as

$$|n\rangle \rightarrow \omega^n |n\rangle , \quad (3.9)$$

so that any n of the form $jN + r$ transforms in the r^{th} irreducible representation of \mathbb{Z}_N . The isotypical components are thus $\mathcal{H}_r = \text{Span}\{|jN + r\rangle\}$ for $j \in \mathbb{Z}$ and $0 \leq r \leq N - 1$. Projectors (3.6) with energy (3.7) have rank n_r in the mod r subspace \mathcal{H}_r of the number basis of Hilbert space.

Note that a single unstable $D9$ -brane is sufficient to generate any number of any type of fractional brane in this simple example. Equivalently, one could have started with N

unstable $D9$ -branes, representing the action of G as a shift operator in the Chan-Paton space, as in [21]. Given that the endpoint of tachyon condensation is universal, we expect these to give equivalent representations of $D7$ -branes on the orbifold, *c.f.* the discussion in section 2.

The ability to represent localized branes on the orbifold in terms of tachyon condensation on a single $D9$ -brane is a general feature of orbifolds without discrete torsion, and follows from the fact that the ‘oscillator’ ground state $|\vec{0}\rangle \in \mathcal{H}$ may be taken to transform in the trivial representation of G (the corresponding projector $|0\rangle\langle 0|$ is the image of a gaussian $2e^{-|z|^2}$ under the map (2.2), and thus invariant under the point group G). The presence of discrete torsion obstructs a construction of fractional branes using a single unstable $D9$ -brane. From (3.3), the action of G on the coordinate operators a_i, a_i^\dagger is unchanged, however the cocycle $\gamma\gamma' = \sigma(\gamma, \gamma') \gamma'\gamma$ must be represented on Hilbert space. The oscillator ground state must therefore lie in a projective representation of G which necessarily has dimension $M > 1$; hence one must start with kM unstable branes.

This requirement of multiple $D9$ -branes is an example of the fact that, if H is N -torsion, then one must have a number of $D9$ branes which is a nonzero multiple of N [13]. In the present case, $H_G^2(Y, U(1)) = H_G^3(Y, \mathbb{Z})$ for equivariant cohomology groups of a finite group G acting on Y . If we take $Y = \mathbb{C}^d$ to be contractible, this boils down to $H^2(G, U(1)) = H^3(G, \mathbb{Z})$; then having a nontrivial element of discrete torsion means there is an H -field turned on and in our case this is a torsion H -field.

3.2. Type IIB

In the $D9\text{-}\overline{D9}$ system, we must construct a set of partial isometries for fractional branes. To begin, let us work with a single brane-antibrane pair. Denote by Σ a G -invariant shift operator of index $-|G|$ with $\bar{\Sigma}\Sigma = 1$ (we called this \bar{T} in equation (2.14), here we reserve that symbol for the fractional brane partial isometry). For example, the $|G|^{\text{th}}$ power of (2.14) will describe a regular representation brane at the origin. Let \mathcal{P}_ρ be the projection onto the isotypical component \mathcal{H}_ρ defined above; then partial isometries for fractional branes are obtained by projecting the partial isometry for the regular representation onto fractional brane subspaces:

$$T_\rho = 1 - \mathcal{P}_\rho + \mathcal{P}_\rho \Sigma \mathcal{P}_\rho . \quad (3.10)$$

The operator T_ρ acts as a shift operator in \mathcal{H}_ρ , and as the identity operator in $\mathcal{H}_{\rho'}$, $\rho' \neq \rho$. These operators satisfy a rather interesting algebra:

$$\begin{aligned}
(T_\rho)^\ell &= 1 - \mathcal{P}_\rho + \mathcal{P}_\rho \Sigma^\ell \mathcal{P}_\rho \\
((T_\rho)^\ell)^\dagger (T_\rho)^\ell &= 1 \\
(T_\rho)^\ell ((T_\rho)^\ell)^\dagger &= 1 - \mathcal{P}_\rho \mathbf{P}_\ell \mathcal{P}_\rho \\
(T_\rho)^\ell (T_{\rho'})^{\ell'} &= (T_{\rho'})^{\ell'} (T_\rho)^\ell = 1 - \mathcal{P}_\rho - \mathcal{P}_{\rho'} + \mathcal{P}_\rho \Sigma^\ell \mathcal{P}_\rho + \mathcal{P}_{\rho'} \Sigma^{\ell'} \mathcal{P}_{\rho'} \quad \rho \neq \rho' .
\end{aligned} \tag{3.11}$$

Here \mathbf{P}_ℓ is a rank ℓ projection operator. From this we can say that the collection of ℓ_ρ fractional branes of type ρ corresponds to the partial isometry

$$T = \prod_{\rho \in \widehat{G}} (T_\rho^{\ell_\rho}) \tag{3.12}$$

The energy is proportional to

$$\text{Tr}_{\mathcal{H}} U(\bar{T}T - 1) + \text{Tr}_{\mathcal{H}} U(T\bar{T} - 1) = U(-1) \sum_{\rho} \ell_\rho \dim(\rho) , \tag{3.13}$$

where $U(-1)$ is the potential at the unstable maximum.

Example: For the orbifold \mathbb{C}/\mathbb{Z}_N , the shift operator is simply $\Sigma = \sum_i |i + N\rangle \langle i| = (a^\dagger \frac{1}{\sqrt{aa^\dagger}})^N$. The isotypical components of \mathcal{H} are again the $r \bmod N$ subspaces \mathcal{H}_r , as in the previous subsection. Although the resulting branes (3.12) carry nontrivial charges in K-theory, and hence are stable in classical open string field theory, the orbifold breaks supersymmetry. In the twisted sectors, there are still massless RR fields coupling to the conserved D-brane charges, but there are also NS-NS tachyons, so that the closed string vacuum is unstable. It would be interesting if one could discern the fate of these charges under closed string tachyon condensation.

Example: The next interesting examples are the ADE orbifolds \mathbb{C}^2/G [21,59]. The matrix $\Gamma \cdot x$ of the ABS construction is simply $\begin{pmatrix} a_1^\dagger & -a_2 \\ a_2^\dagger & a_1 \end{pmatrix}$, on which the rotation group acts by $SU(2)_L \times SU(2)_R$ transformations. Supersymmetry and the symplectic form Θ are preserved for $G \in SU(2)_L$, and the closed string vacuum is stable.

- 1.) A_{N-1} : The cyclic group orbifold $\mathbb{C}^2/\mathbb{Z}_N$ is generated by the element g acting on the raising operators as

$$g(a_1^\dagger, a_2^\dagger) = (\omega a_1^\dagger, \omega^{-1} a_2^\dagger) . \tag{3.14}$$

The isotypical components \mathcal{H}_r of \mathcal{H} are thus $|n_1, n_2\rangle$ with $n_1 - n_2 = r \bmod N$.

- 2.) D_{N+2} : The dihedral group orbifold \mathbb{C}^2/D_{N+2} is generated by (3.14) with $\omega = \exp[i\pi/N]$, as well as

$$h(a_1^\dagger, a_2^\dagger) = (ia_2^\dagger, ia_1^\dagger) , \quad (3.15)$$

which acts on \mathcal{H} as $|n_1, n_2\rangle \rightarrow i^{n_1+n_2}|n_2, n_1\rangle$. The order of the group is $|G| = 4N$.

The isotypical components comprise four sectors of one-dimensional representations

$$\begin{aligned} \mathcal{H}_0^\pm &= \text{Span}\left\{|\ell, 2jN + \ell\rangle \pm |2jN + \ell, \ell\rangle\right\} \quad , \quad j, \ell \in \mathbb{Z}_+ \\ \mathcal{H}_N^\pm &= \text{Span}\left\{|\ell, (2j+1)N + \ell\rangle \pm |(2j+1)N + \ell, \ell\rangle\right\} \quad , \quad j, \ell \in \mathbb{Z}_+ \end{aligned} \quad (3.16)$$

together with $2(N-1)$ sectors based on two-dimensional representations

$$\begin{aligned} \mathcal{H}_r &= \text{Span}\left\{(|\ell, r + 2jN + \ell\rangle, |r + 2jN + \ell, \ell\rangle)\right\} \quad , \\ j, \ell &\in \mathbb{Z}_+ \quad , \quad r = 1, \dots, N-1, N+1, \dots, 2N-1 . \end{aligned} \quad (3.17)$$

- 3.) $\mathcal{T}, \mathcal{O}, \mathcal{I}$: For the group action in the $E_{6,7,8}$ series (tetrahedral, octahedral, icosahedral subgroups of $SU(2)$), the reader may consult for example [59]. The decomposition of \mathcal{H} into its isotypical components is a straightforward if tedious exercise:

$$\mathcal{H}_\rho = \text{Span}\left\{\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \pi_G(g^{-1}) |n_1, n_2\rangle\right\} . \quad (3.18)$$

The structure of the effective field theory is succinctly encoded in a quiver diagram [21,59].

For the ADE series, these are the corresponding extended Dynkin diagrams (see Figure 1).

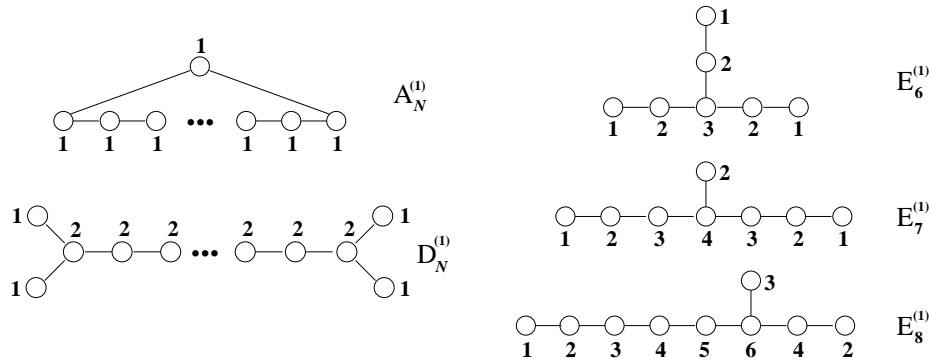


Figure 1. *Dynkin diagrams for affine Lie algebras. The integers attached to each node are the Dynkin labels of the affine root.*

Each irrep is associated to a node of the Dynkin diagram, and has dimension d_ρ equal to the Dynkin label of that node; the ρ representation appears d_ρ times in the regular representation, and thus leads to $U(d_\rho)$ enhanced gauge symmetry for the fractional brane at the origin. In the present context, the nodes label isotypical components of the Hilbert space, *e.g.* (3.16) on the corners of the D_N diagram and (3.17) for the chain of two-dimensional representations. A general quiver representation consists of n_ρ fractional branes of type ρ . The links of the diagram specify the bifundamental matter content, which arises from components of the gauge field components A_i along the orbifold. A link joining representations ρ and ρ' has multiplicity $n_\rho n_{\rho'}$ in the low energy Lagrangian. It is important to note that, if n_ρ is zero on some node, then that means the matter fields A_z in question make open strings with one end on a fractional brane, and another end ‘on the closed string vacuum’. These are precisely the states that obtained string scale mass due to the Higgs mechanism in [12], and hence are not reliably treated in the low energy approximation; as discussed in section 2.2, they are argued generally to be removed by stringy effects not visible at the level of the effective theory.

3.3. Translating branes away from the orbifold point

The generic fractional brane has no moduli space corresponding to translations away from the orbifold point; for instance, in the BPS case it carries the charges of a bound state of branes wrapping various collapsed cycles at the fixed point [21,24,23], thus it is pinned there. Indeed, as we just argued, a single fractional brane ($\ell_\rho = 1$, and $\ell_{\rho'} = 0$ for $\rho' \neq \rho$) has no light scalar field in its effective action that could serve as a Goldstone mode. However, a brane in the regular representation is free to move away. Indeed, a brane localized at a generic point on the orbifold consists of a G -orbit of such branes on the covering space; the orbit *is* the regular representation on the coordinates. Again, the fact that the noncommutative framework reproduces the correct low-energy dynamics guarantees that the moduli space will be the same as the commutative description; one can simply give an expectation value to the transverse scalars A_i to move a regular representation brane away from the origin, as in section 2.3. Thus, regular representation branes away from the orbifold point are special cases of (2.17), (2.19) for points arrayed along a G -orbit.

To illustrate the presence of a moduli space for the regular representation brane, and the absence of translational moduli for fractional branes, consider the \mathbb{C}/\mathbb{Z}_N orbifold.

Implement the orbifold projection on the algebra $\mathcal{A} \otimes \text{Mat}_N$ via the projection $\mathcal{O} = \mathcal{U} \mathcal{O} \mathcal{U}^{-1}$, with

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \gamma \\ \gamma & 0 & & & \vdots \\ 0 & \gamma & \ddots & & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & \gamma & 0 \end{pmatrix}, \quad (3.19)$$

where γ acts in \mathcal{H} as usual by $\gamma|z\rangle = |\omega z\rangle$. An invariant tachyon projector for the regular representation translated away from the origin is

$$\mathbf{1} - T = \begin{pmatrix} |z\rangle\langle z| & 0 & \cdots & 0 \\ 0 & |\omega z\rangle\langle \omega z| & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & |\omega^{-1}z\rangle\langle \omega^{-1}z| \end{pmatrix}; \quad (3.20)$$

the corresponding gauge field is⁵

$$A_z = \begin{pmatrix} z\mathbf{1} & 0 & \cdots & 0 \\ 0 & \omega z\mathbf{1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \omega^{-1}z\mathbf{1} \end{pmatrix}. \quad (3.21)$$

Clearly $\text{Tr}[A_z, T]^2$ vanishes, and z parametrizes motion of the brane on the orbifold. On the other hand, a generating set of fractional brane projectors is

$$T_{ab}^{(\ell)} = \frac{1}{N} \omega^{\ell(a-b)} |\omega^a z\rangle\langle \omega^b z|, \quad (3.22)$$

for $\ell = 0, \dots, N-1$; here $a, b = 0, \dots, N-1$ are Chan-Paton indices. For a fractional brane $T = \sum_{\ell} \varepsilon_{\ell} T^{(\ell)}$ with $\varepsilon_{\ell} = 0, 1$, one readily verifies that

$$\text{Tr}[A, T]^2 = |z|^2 \langle \vec{\varepsilon}, \vec{\varepsilon} \rangle \quad (3.23)$$

where $\langle \cdot, \cdot \rangle$ is the inner product with respect to the Cartan matrix of the affine Lie algebra \hat{A}_{N-1} . Thus the attempt to excite transverse motion of fractional branes costs string scale energy, verifying our claim that such excitations are among those that should be dropped from a proper low-energy description. On the other hand, superposing all the fractional brane projectors recovers the regular representation soliton (3.20) (the off-diagonal elements cancel due to \mathbb{Z}_N phases); correspondingly, the vector $\varepsilon_{\ell} = 1$ for all $\ell = 0, \dots, N-1$ is precisely the zero norm vector for the inner product (3.23), so that the expectation value of A_z parametrizes a flat direction.

⁵ Note that for $A_z = 0$, the projector (3.20) is unitarily equivalent to the standard projector $T = \mathbf{1} - \mathbf{P}_N$ built from (2.17); note also that the physical displacement of the branes involves the *relative* orientation of the tachyon and gauge fields in $\text{Mat}_N(\mathcal{A})$.

4. Compact orbifolds

We described in the previous section the ingredients for constructing D-branes on the orbifold \mathbb{R}^{2d}/G in terms of the crossed product $\mathcal{A} \rtimes G$ of the Moyal algebra of functions \mathcal{A} on the noncommutative hyperplane \mathbb{R}^{2d} with the orbifold group G . In fact this prescription provides a procedure to obtain the D-branes on a general orbifold. Given an algebra $\mathcal{C}(\mathcal{Y})$ of functions on a (possibly noncommutative) space \mathcal{Y} , and carrying a group action $\alpha : G \rightarrow \text{Aut}(\mathcal{C}(\mathcal{Y}))$, the crossed product $\mathcal{C}(\mathcal{Y}) \rtimes G$ constructs the orbifold. The physical string modes are again the commutant of G in the crossed product.

4.1. The torus as $\mathbb{R}^d/\mathbb{Z}^d$

As a first example, let us construct the algebra of functions $\mathcal{A}_\theta(\mathbf{T}^2)$ on the two-dimensional noncommutative torus as the commutant of \mathbb{Z}^2 in the crossed product $\mathcal{A}(\mathbb{R}^2) \rtimes \mathbb{Z}^2$ (this has been discussed in [60], section 2). The generators U, V of the algebra \mathcal{A}_θ satisfy the relation

$$VU = e^{2\pi i \theta} UV . \quad (4.1)$$

Let $\mathcal{T}(m\omega_1 + n\omega_2)$ span the algebra of translation operators (2.3) representing translations along the basic periods of the torus. Elements of $\mathcal{A}(\mathbb{R}^2) \rtimes \mathbb{Z}^2$ are written

$$\sum_{m,n \in \mathbb{Z}} a_{mn} g_{mn} \quad (4.2)$$

for a_{mn} in the Moyal algebra of functions on \mathbb{R}^2 and $g_{mn} \in \mathbb{Z}^2$, i.e. $g_{mn}g_{pq} = g_{m+p,n+q}$. The crossed product algebra is

$$\sum_{mn} a_{mn} g_{mn} \cdot \sum_{pq} b_{pq} g_{pq} = \sum_{mnkl} a_{mn} \alpha_{g_{mn}}(b_{pq}) g_{m+p,n+q} , \quad (4.3)$$

where the group \mathbb{Z}^2 is taken to act on $\mathcal{A}(\mathbb{R}^2)$ via conjugation by translations (2.3)

$$\alpha_{g_{mn}}(\mathcal{O}) = \mathcal{T}(m\omega_1 + n\omega_2) \mathcal{O} \mathcal{T}(-m\omega_1 - n\omega_2) . \quad (4.4)$$

Now let ω'_i $i = 1, 2$ be such that $\omega'_i \bar{\omega}'_j - \bar{\omega}'_i \omega'_j = 2\pi i \delta_{ij}$. Then from (2.3) the commutant of $G = \mathbb{Z}^2$ in $\mathcal{A}(\mathbb{R}^2) \rtimes \mathbb{Z}^2$ is spanned by elements of the form $\mathcal{T}(p\omega'_1 + q\omega'_2)g_{rs}$. In other words, we identify $U = \mathcal{T}(\omega'_1)$, $V = \mathcal{T}(\omega'_2)$, and $\theta = \omega'_1 \bar{\omega}'_2 - \bar{\omega}'_1 \omega'_2$, and the commutant is $\mathcal{A}_\theta \times G$. As in all examples of abelian orbifolds, the commutant of G in the crossed

product is the collection of invariant open string states we want, times a copy of functions on the group (which is equivalent to Chan-Paton structure).

Since the subalgebra U^j is commutative, we can alternatively orbifold first by the action of one \mathbb{Z} factor, and regard the algebra \mathcal{A}_θ as $\mathcal{C}(\mathbf{T}) \rtimes \mathbb{Z}$. This means there will be representations of \mathcal{A}_θ on the Hilbert space $L^2(\mathbf{T})$, by clock and shift operators. All these different algebraic descriptions of the noncommutative torus – $\mathcal{A}(\mathbb{R}^2) \rtimes \mathbb{Z}^2$, $\mathcal{C}(\mathbf{T}) \rtimes \mathbb{Z}$, and \mathcal{A}_θ – are equivalent [56].

The story for higher-dimensional noncommutative tori is quite similar [60]. The d -dimensional noncommutative torus algebra \mathcal{A}_Θ is generated by translation operators U_i , $i = 1, \dots, d$, with relations $U_i^* = U_i^{-1}$ and $U_i U_j = \exp[2\pi i \Theta^{ij}] U_j U_i$, where $\Theta^t = -\Theta$. This algebra will arise as the commutant of the algebra of translations by a lattice on the noncommutative plane with basis vectors ω_i . The generator of translations is $\partial_\alpha = -i\Theta_{\alpha\beta}^{-1}[x^\beta, \cdot]$, so lattice translations are implemented by

$$\mathcal{T}_{\vec{n}} = \exp[n^i \omega_i^\alpha \partial_\alpha] . \quad (4.5)$$

Repeating the same steps as above, the commutant of the $\mathcal{T}_{\vec{n}}$ is spanned by

$$U_{\vec{m}} = \exp[2\pi i m_j \tilde{\omega}_\alpha^j x^\alpha] , \quad (4.6)$$

where $\tilde{\omega}^j$ are basis vectors for the dual lattice (momentum space). Thus the U_i are generators of translations on the lattice of momenta on the torus. The algebra \mathcal{A}_Θ has a grading by \mathbb{Z}^d , and the vector space basis for the algebra $U_{\vec{m}}$, $\vec{m} \in \mathbb{Z}^d$, has multiplication

$$\begin{aligned} U_{\vec{m}} U_{\vec{n}} &= U_{\vec{m}+\vec{n}} e^{2\pi i m_i \Gamma^{ij} n_j} \\ U_{\vec{n}} U_{\vec{m}} &= U_{\vec{m}} U_{\vec{n}} e^{2\pi i n_i \Theta^{ij} m_j} . \end{aligned} \quad (4.7)$$

By a redefinition $U_{\vec{m}} \rightarrow \exp[2\pi i \vec{m} \cdot M \cdot \vec{m}] U_{\vec{m}}$, where M is a symmetric matrix, we can shift away the symmetric part of Γ . The antisymmetric part is constrained to be $\Gamma - \Gamma^t = \Theta$; in particular, one could take $\Gamma = \frac{1}{2}\Theta$. The general element of the algebra is a linear combination of the $U_{\vec{n}}$ with coefficients rapidly decreasing for large $|\vec{n}|$. (The C^* algebra is then obtained by taking the norm completion.)

4.2. String field action

Now let us consider the truncation of the string field theory action which was the starting point for the discussion of [12] in the noncompact case. The tachyon fields and hence the Lagrangian density should be functions from \mathcal{X} to the algebra \mathcal{A}_Θ . Moreover, we expect that the Lagrange density should have the same form as in (2.5). Since the grading on the algebra by \mathbb{Z}^d is a grading by momenta, the action must be given by the *trace* on the algebra \mathcal{A}_Θ , defined by

$$\tau\left(\sum_{\vec{n}\in\mathbb{Z}^d} a_{\vec{n}} U_{\vec{n}}\right) := a_{\vec{0}} \quad (4.8)$$

Therefore, with the simple replacement of the Hilbert space trace in (2.5) by the trace on \mathcal{A}_Θ we obtain the action for compactification on the torus (for IIA; IIB is similar):

$$S = \frac{c}{G_s} \int_{\mathcal{X}} d^{10-d}x \sqrt{G} \tau \left[\frac{1}{2} f(T^2 - 1) G^{\mu\nu} D_\mu T D_\nu T - V(T^2 - 1) \right. \\ \left. - \frac{1}{4} h(T^2 - 1) F_{\mu\nu} F^{\mu\nu} + \dots \right]. \quad (4.9)$$

Here we use a derivative that acts on the algebra as $\partial_i U_j = 2\pi i \delta_{ij} U_j$, motivated by the above construction of the torus algebra from the quotient of the noncommutative plane. Technically speaking, we should also specify a projective module (the analogue of a vector bundle for noncommutative algebras) for which the gauge field provides a connection; for simplicity, we take the free module \mathcal{A}_Θ^N provided by the algebra itself, corresponding to a topologically trivial connection on the worldvolume theory of N noncommutative D9-branes.

The action (4.9) arises in the zero-slope limit (*c.f.* [1,2]) of string dynamics on the torus in the presence of a B field. In this limit, the closed string metric $g_{ij} \rightarrow 0$, so closed string modes of nonzero momentum decouple. One might worry that the limit is afflicted by a tower of closed string winding modes descending to zero mass; however, the term $\alpha'^2 w^i (B g^{-1} B)_{ij} w^j$ in their mass squared remains finite (and of course, they only couple to the dynamics at the quantum level). A somewhat more general noncommutative limit, $\alpha' B \rightarrow \infty$ at fixed g , was considered in [12]. One might wonder whether this limit is valid on the torus, where the moduli space is identified under shifts of $\alpha' B$ by integers. The point is that the action (4.9) refers to a particular (free) module, and one cannot shift away a large B without changing the module to another one with large lower brane charge (whose physics would of course be equivalent). If we fix the brane we have before tachyon condensation, then the meaning of large B on that brane is unambiguous.

4.3. D-brane projectors on T^d

According to the action of the previous subsection the non-BPS branes on the noncommutative torus should again be described, in the zero-slope or Seiberg-Witten limit, by projection operators [27,28,29]. In general, the range of a projection operator \mathbf{P} describing a collection of noncommutative D-brane solitons is the Chan-Paton space of those branes; the algebra $\mathbf{P}\mathcal{A}\mathbf{P}$ is the algebra of endomorphisms of the Chan-Paton space. In the non-compact case of the noncommutative plane, this space is finite (say N) dimensional, and the endomorphism algebra is the algebra of $N \times N$ matrices. In the torus case, due to the infinite collection of image branes, the endomorphism algebra should be more complicated. Consider the two-dimensional case. Start with an unstable D9-brane on a noncommutative torus associated to the algebra \mathcal{A}_θ , and condense the tachyon to $T = \mathbf{1} - \mathbf{P}$; for some appropriate projector \mathbf{P} , we expect to be able to describe an unstable D7-brane on the torus. But by T-duality, the algebra on the D7-brane must be Morita equivalent to the algebra \mathcal{A}_θ we started with – the T-duality that inverts the torus interchanges D7-branes and D9-branes, and sends $\theta \rightarrow -1/\theta$ [1,5,6,7,8,9,2,48]. Thus we should have an isomorphism of C^* -algebras

$$\mathbf{P}\mathcal{A}_\theta\mathbf{P} = \mathcal{A}_{\theta'},$$

for $\theta' = -1/\theta$. Let us see that this is so. First, it is true [61] that for any C^* algebra \mathcal{A} and projector \mathbf{P} on \mathcal{A} , $\mathbf{P}\mathcal{A}$ is an $\mathcal{A}\mathbf{P}\mathcal{A} - \mathbf{P}\mathcal{A}\mathbf{P}$ Morita equivalence bimodule.⁶ Now note that $\mathcal{A}\mathbf{P}\mathcal{A}$ is a two-sided ideal of \mathcal{A} . However, \mathcal{A}_θ is a simple algebra [62], meaning it has no proper two-sided ideals; therefore $\mathcal{A}_\theta\mathbf{P}\mathcal{A}_\theta = \mathcal{A}_\theta$. We conclude that $\mathbf{P}\mathcal{A}_\theta\mathbf{P}$ and \mathcal{A}_θ are Morita equivalent. It is a fact [63] that the only C^* algebras with identity that are Morita equivalent to \mathcal{A}_θ are the algebras $M_n(\mathcal{A}_{\theta'})$, where θ' is related to θ by a fractional linear transformation, $\theta' = \frac{a\theta+b}{c\theta+d}$. Thus the endomorphisms of the Chan-Paton bundle on the lower brane constitute a noncommutative torus algebra T-dual to the original one. To complete the identification, we need only to determine the solitonic D-brane numbers of a given projector via the above T-duality relation.

⁶ An $\mathcal{A}-\mathcal{B}$ Morita equivalence bimodule \mathcal{M} is a left \mathcal{B} -module and a right \mathcal{A} -module, equipped with \mathcal{A} -valued and \mathcal{B} -valued inner products $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ such that $\langle f, g \rangle_{\mathcal{B}} h = f \langle g, h \rangle_{\mathcal{A}}$, for $f, g, h \in \mathcal{M}$. The existence of the bimodule establishes the Morita equivalence of \mathcal{A} and \mathcal{B} ; see for instance [61].

Projection operators in the two-dimensional noncommutative torus algebra can be given explicitly [64,65]. Assume (without loss of generality) $\frac{1}{2} < \theta < 1$, and let f and g be periodic functions such that

$$f(e^{2\pi i x}) = \begin{cases} \text{a smooth function increasing from 0 to 1} & x \in [0, 1 - \theta] \\ 1 & x \in [1 - \theta, \theta] \\ 1 - f(e^{2\pi i(x-\theta)}) & x \in [\theta, 1] \end{cases} \quad (4.10)$$

$$g(e^{2\pi i x}) = \begin{cases} 0 & x \in [0, \theta] \\ \sqrt{f - f^2} & x \in [\theta, 1] \end{cases}$$

Then

$$\mathbf{P}_\theta = g(V)U + f(V) + U^{-1}\bar{g}(V) \quad (4.11)$$

is a projector (constructed by Powers and Rieffel [64]) which, together with the trivial projector $\mathbf{1}$, generates $K_0(\mathcal{A}_\theta)$ when θ is irrational. The use of this projector to construct D-branes on tori was discussed in [27,28,29].

Projection operators enable us to construct lower-dimensional non-BPS branes as solitons in higher-dimensional non-BPS branes. Solitonic field configurations on the torus carrying RR charge can also be constructed from projection operators. The discussion involves yet more machinery of noncommutative geometry, and is therefore deferred to [30]. We will see there that the Powers-Rieffel projector (4.11) may be used to construct the full K-theory lattice of brane bound states on the two-dimensional torus. In anticipation of this development (and to explore more fully the possible unstable brane configurations), we recall a construction of a complete set of projectors $\mathbf{P}_{n+m\theta}$ by Rieffel [64]. Note that a projector for Dp - $D(p-2)$ charges (r, s) with $0 < r + s\theta < 1$ is built as follows [64]: Let $\mathcal{C}(\mathbf{T})$ be the functions on the circle. The Powers-Rieffel projector (4.11) uses such functions to construct the projector for $(r, s) = (0, 1)$ or $(1, -1)$ depending on conventions, with $\theta < 1$. But $\mathcal{C}_m(\mathbf{T})$, the functions of period $1/m$, is a subalgebra of $\mathcal{C}(\mathbf{T})$, and on $\mathcal{C}_m(\mathbf{T})$, a shift by θ (the action of V in the noncommutative torus algebra (4.1)) looks like a shift in $\mathcal{C}(\mathbf{T})$ by $\{m\theta\}$, the fractional part of $m\theta$. Now just repeat the construction of the projector (4.11) using this function space; if the original projector associated to θ had, say, quantum numbers $(r, s) = (0, 1)$, then the new one has quantum numbers $r = n$, $s = m$, such that $0 < n + m\theta < 1$. One can add Chan-Paton structure, put this projector in the first diagonal entry, and the trivial projector $\mathbf{1}$ in the rest, to get a projector with any $n + m\theta > 0$. These are precisely the stable (real, as oppose to virtual) bundles in

the K-group of the noncommutative 2-torus, which is isomorphic to $\mathbb{Z} + \mathbb{Z}\theta$ for irrational θ . The above construction is unique up to isomorphism; it is a theorem [60] that (in any dimension), for θ irrational, any two projective modules representing the same element of K-theory are isomorphic.

We can now canonically associate particular projectors $\mathbf{P}_{n+m\theta}$ to the appropriate collection of unstable branes with brane numbers (m, n) .⁷ Recall that the normalization of the trace transforms as [1,5,6,7,8,9,2,48]

$$\tau_{\theta'}[\] = (c\theta + d)^{-1} \tau_{\theta}[\] \quad (4.12)$$

under the Morita equivalence that maps $\theta \rightarrow \theta' = \frac{a\theta+b}{c\theta+d}$. Since $\tau(\mathbf{P}_{\theta}) = \theta$, we conclude that the trace of the identity in $\mathcal{A}_{\theta'}$ is correctly reproduced by the trace of \mathbf{P}_{θ} if the algebra $\mathcal{A}_{\theta'} = \mathbf{P}_{\theta}\mathcal{A}_{\theta}\mathbf{P}_{\theta}$ has $\theta' = -1/\theta$; indeed this is the T-duality that maps D7-branes to D9-branes. Similarly, for the algebra on the range $\mathcal{A}_{\theta''} = (\mathbf{1} - \mathbf{P}_{\theta})\mathcal{A}_{\theta}(\mathbf{1} - \mathbf{P}_{\theta})$, one finds $\theta'' = \frac{-1}{1-\theta}$, which correctly maps $(D7, D9)$ charges $(-1, 1)$ to $(0, 1)$. More generally, $\mathcal{A}_{\hat{\theta}} = \mathbf{P}_{n+m\theta}\mathcal{A}\mathbf{P}_{n+m\theta}$ is associated to brane numbers (m, n) via the fractional linear transformation $\hat{\theta} = \frac{a\theta+b}{m\theta+n}$ that transforms the brane numbers to $(0, 1)$.

The potential energy term in (4.9) is proportional to the trace (4.8) of the projector $\tau(\mathbf{P}_{n+m\theta})$, and this is the leading term in the energy in the zero-slope or Seiberg-Witten limit.⁸ A simple calculation yields $\tau(\mathbf{P}_{n+m\theta}) = n + m\theta$.⁹ Thus N unstable D9-branes can decay into any ensemble of unstable branes such that the corresponding projector has trace $n + m\theta < N$. In principle, this might lead to an infinite collection of arbitrarily finely spaced critical points in the tachyon potential, for instance of a single unstable D9-brane, corresponding to successively better rational approximations $-m/n \sim \theta$. Considerations of the kinetic energy terms in (4.9) could reduce the number of critical points to a finite number. Although these critical points are closely spaced in energy, nearby ones may

⁷ Note that in the IIA case, the K-theory classes of the projectors $\mathbf{P}_{n+m\theta}$ are not measuring conserved charges of the brane configuration. This is because there is a continuous path of nonsingular configurations $T = s\mathbf{P}_{n+m\theta} + (1-s)\mathbf{P}_{n'+m'\theta}$, $0 \leq s \leq 1$, that smoothly interpolates between any two of them. Instead, in this case n, m characterize the various critical points of the action.

⁸ The trace gives a measure of the dimension of the space $\mathbf{P}_{n+m\theta}\mathcal{A}_{\theta}\mathbf{P}_{n+m\theta}$ thought of as an \mathcal{A}_{θ} module, as well as the brane tension; *c.f.* [1,66].

⁹ The trace is a cohomological invariant on the algebra, and it provides the map from the lattice of K-theory charges to the ordered group $\mathbb{Z} + \mathbb{Z}\theta$ mentioned above.

correspond to widely different field configurations (rather similar to the irrational axion [67]), with a corresponding large potential barrier. Each of these configurations is of course unstable toward decay to the closed string vacuum.

For orbifold applications, \mathbf{P}_θ can only be used for \mathbb{Z}_2 quotients, since its invariance group is $U \rightarrow U^{-1}$, $V \rightarrow V^{-1}$; the Powers-Rieffel projector (4.11) is ill-suited for the construction of more general orbifolds which act by permuting the generators U and V (see below). A general procedure to construct projection operators on \mathcal{A}_θ uses Morita equivalence. We begin with the projective modules for \mathcal{A}_θ introduced by Connes in [31]. In the simplest example the module is the Schwarz space $\mathcal{S}(\mathbb{R})$ of smooth functions of rapid decrease at infinity. In fact, this module provides a Morita equivalence bimodule \mathcal{M} between $\mathcal{A} = \mathcal{A}_\theta$ and $\mathcal{B} = \mathcal{A}_{1/\theta}$ as follows [31,63]. Let $VU = \lambda UV$, $\lambda = e^{2\pi i\theta}$, and $\tilde{V}\tilde{U} = \mu\tilde{U}\tilde{V}$, $\mu = e^{2\pi i/\theta}$. We have left and right actions of \mathcal{B} and \mathcal{A} , respectively, on $f \in \mathcal{S}(\mathbb{R})$ via

$$\begin{aligned} (fV)(t) &= e^{2\pi it} f(t) & , & & (fU)(t) &= f(t + \theta) \\ (\tilde{V}f)(t) &= e^{-2\pi it/\theta} f(t) & , & & (\tilde{U}f)(t) &= f(t + 1) . \end{aligned} \quad (4.13)$$

Then for functions $f, g \in \mathcal{S}(\mathbb{R})$ we can define \mathcal{A} - and \mathcal{B} -valued inner products

$$\begin{aligned} \langle f, g \rangle_{\mathcal{A}} &= \sum_{m,n} \langle f, g \rangle_{\mathcal{A}}(m, n) \cdot U^m V^n \\ \langle f, g \rangle_{\mathcal{A}}(m, n) &= \theta \int_{-\infty}^{\infty} \overline{f(t + m\theta)} g(t) e^{2\pi i(-nt)} dt \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} \langle f, g \rangle_{\mathcal{B}} &= \sum_{m,n} \langle f, g \rangle_{\mathcal{B}}(m, n) \cdot \tilde{U}^m \tilde{V}^n \\ \langle f, g \rangle_{\mathcal{B}}(m, n) &= \int_{-\infty}^{\infty} f(t - m) \overline{g(t)} e^{2\pi i(nt/\theta)} dt . \end{aligned} \quad (4.15)$$

One can show ([60], section 2) that

$$\langle f, g \rangle_{\mathcal{B}} h = f \langle g, h \rangle_{\mathcal{A}}, \quad (4.16)$$

which is the key statement of Morita equivalence. In particular, the norm completion of $\mathcal{M}_{ab} \otimes_{\mathcal{B}} \mathcal{M}_{ba}$ is \mathcal{A} ,¹⁰ and of $\mathcal{M}_{ba} \otimes_{\mathcal{A}} \mathcal{M}_{ab}$ is \mathcal{B} .¹¹ One constructs nontrivial modules by

¹⁰ The notation $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}$ stands for the bimodule spanned by elements of the form $m \otimes n$ subject to the relations that $mb \otimes n = m \otimes bn$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$, $b \in \mathcal{B}$.

¹¹ In string terms, this means that sewing open strings with boundary conditions ab and ba yields the full algebra of aa strings or bb strings, depending on which ends are sewn. The fact that $\mathcal{M}_{ba} \otimes_{\mathcal{A}} \mathcal{M}_{ab}$ is the entire algebra \mathcal{A}_{bb} follows from the generalized Cardy condition.

picking suitable functions f_i , $i = 1 \dots N$ such that $\sum_i \langle f_i, f_i \rangle_{\mathcal{B}} = \mathbf{1}_{\mathcal{B}}$; then $\langle f_i, f_j \rangle_{\mathcal{A}}$ is a nontrivial projector in $\text{Mat}_N(\mathcal{A})$.¹² In particular, [65] constructs a projection operator homotopic to (4.11) starting from the Schwarz function $f = e^{-\pi t^2/\theta}$. It turns out (and this is nontrivial) that $B = \langle f, f \rangle_{\mathcal{B}}$ is invertible. It then easily follows that

$$\mathbf{P}_{\theta} = \langle B^{-1/2} f, B^{-1/2} f \rangle_{\mathcal{A}} \quad (4.17)$$

is a projector. Since the projector is built out of the gaussian function, it is clearly invariant under the \mathbb{Z}_4 operation of Fourier transformation in $\mathcal{S}(\mathbb{R})$, which sends $U \rightarrow V$, $V \rightarrow U^{-1}$. For special rational values $\theta = 1/q$, $q \in \mathbb{Z}$, one can find explicit expressions for \mathbf{P}_{θ} in terms of theta functions of U and V [65]. We give a general expression below.

The above elegant constructions, due to Connes and Rieffel [31,63,60], of the projective modules and their Morita equivalence properties generalizes beautifully to higher dimensional tori, in terms of a representation on $L^2(\mathbb{R}^p \times \mathbb{Z}^q \times F)$, where F is a finite group and $2p + q = d$. We give a brief summary of it in appendix A. (This construction is also reviewed in [41], and interpreted physically in [2].)

Since there are exact solutions (2.12) for the noncommutative plane, one might hope that one can find a compatible gauge field such that the tachyon and gauge field equations of motion on the torus are solved beyond leading order in the limit of large noncommutativity. We will now give a partial solution to this problem. The tachyon field equations will be solved exactly if we can find a compatible connection such that $DT = 0$. The above bimodule construction of projectors is helpful in this regard by allowing us to find such a connection. Some useful identities for the bimodule \mathcal{M} are

$$\begin{aligned} \langle f, g \rangle_{\mathcal{A}}^* &= \langle g, f \rangle_{\mathcal{A}} & \langle f, g \rangle_{\mathcal{B}}^* &= \langle g, f \rangle_{\mathcal{B}} \\ \langle f, ga \rangle_{\mathcal{A}} &= \langle f, g \rangle_{\mathcal{A}} a & \langle bf, g \rangle_{\mathcal{B}} &= b \langle f, g \rangle_{\mathcal{B}} \\ \langle fa^*, g \rangle_{\mathcal{A}} &= a \langle f, g \rangle_{\mathcal{A}} & \langle f, bg \rangle_{\mathcal{B}} &= \langle f, g \rangle_{\mathcal{B}} b^* , \end{aligned} \quad (4.18)$$

where $f, g \in \mathcal{M}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$. The derivative acts as (for $\Phi = -\Theta^{-1}$) [31]¹³

$$\begin{aligned} (d_1 f)(t) &= -\frac{2\pi i}{\theta} t f(t) \\ (d_2 f)(t) &= \frac{d}{dt} f(t) ; \end{aligned} \quad (4.19)$$

¹² The module just constructed is that of $p(p-2)$ strings, *c.f.* [6,2]. Thus it naturally relates the trivial projector for, say, the algebra of $(p-2)$ -($p-2$) strings, to a nontrivial projector for the algebra of p - p strings. Indeed, Poisson resummation is a key ingredient both in the demonstration [60] of Morita equivalence of \mathcal{A}_{θ} algebras and in string T-duality.

¹³ Warning: There are some incompatible factors of 2π between standard conventions for the noncommutative plane and torus, *c.f.* (2.6).

this reproduces the derivation $\partial_i U_j = 2\pi i \delta_{ij} U_j$ on the algebra provided we identify

$$\begin{aligned}\partial_i(\langle f, g \rangle_{\mathcal{A}}) &= \langle d_i f, g \rangle_{\mathcal{A}} + \langle f, d_i g \rangle_{\mathcal{A}} \\ \partial_i(\langle f, g \rangle_{\mathcal{B}}) &= -\theta(\langle d_i f, g \rangle_{\mathcal{B}} + \langle f, d_i g \rangle_{\mathcal{B}}) .\end{aligned}\tag{4.20}$$

Note that from the bimodule property $\langle f, g \rangle_{\mathcal{B}} h = f \langle g, h \rangle_{\mathcal{A}}$, if there exists \mathbf{f} with $\langle \mathbf{f}, \mathbf{f} \rangle_{\mathcal{B}} = \mathbf{1}$ and $\langle \mathbf{f}, \mathbf{f} \rangle_{\mathcal{A}} = \mathbf{P}$, then $h = \mathbf{f} \langle \mathbf{f}, h \rangle_{\mathcal{A}}$ for all h , in particular for $h = d\mathbf{f}$. Then it is straightforward to verify that

$$\begin{aligned}\mathbf{P} \langle d\mathbf{f}, \mathbf{f} \rangle_{\mathcal{A}} &= -\langle \mathbf{f}, d\mathbf{f} \rangle_{\mathcal{A}} \mathbf{P} \\ \langle d\mathbf{f}, \mathbf{f} \rangle_{\mathcal{A}} \mathbf{P} &= \langle d\mathbf{f}, \mathbf{f} \rangle_{\mathcal{A}} \\ \mathbf{P} \langle \mathbf{f}, d\mathbf{f} \rangle_{\mathcal{A}} &= \langle \mathbf{f}, d\mathbf{f} \rangle_{\mathcal{A}} ,\end{aligned}\tag{4.21}$$

from which it follows that $d\mathbf{P} + A\mathbf{P} - \mathbf{P}A = 0$ for the connection

$$A = \langle \mathbf{f}, d\mathbf{f} \rangle_{\mathcal{A}} - \langle d\mathbf{f}, \mathbf{f} \rangle_{\mathcal{A}} .\tag{4.22}$$

Note that, because it is not constructed using this bimodule procedure, it is not clear whether the Powers-Rieffel projector (4.11) admits a compatible connection such that the tachyon field equations are solved exactly; therefore, it may only be a solution to the field equations in the leading order of the limit of large $\alpha' B$.

It remains to find the function \mathbf{f} among the class with $\langle \mathbf{f}, \mathbf{f} \rangle_{\mathcal{B}} = \mathbf{1}$ that minimizes the gauge field kinetic energy. This is the step we have not carried out. The gauge kinetic energy in (2.5) is proportional to

$$\tau \left[\mathbf{P} (dA + [A, A])^2 \right] .\tag{4.23}$$

We have not succeeded in minimizing this energy.

As an aside, a rather explicit formula for the projector (4.17) may be given. Let $f(t) = e^{-at^2 - bt}$ with a real and positive. The \mathcal{B} algebra acts as

$$(\tilde{U}^m \tilde{V}^n f)(t) = f(t) e^{-(2ma + \frac{2\pi i n}{\theta})t} e^{-am^2 - bm - \frac{2\pi i m n}{\theta}}\tag{4.24}$$

Thus, a gaussian goes into a gaussian up to a linear exponential in t and a prefactor. Thus if we define $B := \langle f, f \rangle_{\mathcal{B}}$ then (simplifying by taking $b = 0$)

$$B^{-1/2} f = \left(1 + \sum_{k=1}^{\infty} \frac{\Gamma(k-1/2)}{\Gamma(-1/2)k!} \left(\frac{\pi}{2a} \right)^{k/2} \sum_{m_k, n_k}' e^{-Q_k(m, n)} \prod_{i=1}^k e^{-(2m_i a + 2\pi i n_i / \theta)t} \right) e^{-at^2}\tag{4.25}$$

where the sum \sum' is over tuples $(m_1, n_1), \dots, (m_k, n_k)$ of nonzero pairs of integers (m, n) and $Q_k(m, n)$ is a quadratic form:

$$Q_k(m, n) = \frac{a}{2} \left(\sum_{i=1}^k m_i^2 \right) + a \left(\sum_i m_i \right)^2 + \frac{\pi^2}{2a\theta^2} \left(\sum_{i=1}^k n_i^2 \right) + \frac{i\pi}{\theta} \left(\sum_i n_i m_i + 2 \sum_{i>j} n_i m_j \right). \quad (4.26)$$

4.4. T^d/G : torus orbifolds

The torus is the orbifold of the plane by a lattice, and the construction of D-brane solitons as projection operators parallels the noncompact case. If the Narain data of a torus is left invariant by some subgroup \mathcal{T}_{fix} of the T-duality group $\mathcal{T} = O(d, d; \mathbb{Z})$ of a d -dimensional noncommutative torus, one may further orbifold by \mathcal{T}_{fix} . The T-duality group \mathcal{T} consists of transformations

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \quad \mathbf{a}^t \mathbf{c} + \mathbf{c}^t \mathbf{a} = 0, \quad \mathbf{b}^t \mathbf{d} + \mathbf{d}^t \mathbf{b} = 0, \quad \mathbf{a}^t \mathbf{d} + \mathbf{c}^t \mathbf{b} = \mathbf{1}, \quad (4.27)$$

acting on the closed string Narain data $E = g + 2\pi\alpha' B$ via

$$\begin{aligned} E &\longrightarrow (\mathbf{a}E + \mathbf{b})(\mathbf{c}E + \mathbf{d})^{-1} \\ g_s &\longrightarrow g_s \sqrt{\det[\mathbf{c}E + \mathbf{d}]} \end{aligned} \quad (4.28)$$

In terms of the noncommutative parametrization of the Narain data [7,8,9,2,48]

$$\frac{1}{g+B} = \Theta + \frac{1}{G+\Phi}, \quad (4.29)$$

the T-duality transformation (4.28) is associated to the Morita equivalence of the algebra \mathcal{A}_Θ , equation (4.7), to its images under $O(d, d)$ transformations

$$\begin{aligned} \Theta &\longrightarrow (\mathbf{c} + \mathbf{d}\Theta)(\mathbf{a} + \mathbf{b}\Theta)^{-1} \\ G &\longrightarrow (\mathbf{a} + \mathbf{b}\Theta)G(\mathbf{a} + \mathbf{b}\Theta)^t \\ \Phi &\longrightarrow (\mathbf{a} + \mathbf{b}\Theta)\Phi(\mathbf{a} + \mathbf{b}\Theta)^t + \mathbf{b}(\mathbf{a} + \mathbf{b}\Theta)^t \\ G_s &\longrightarrow G_s (\det[\mathbf{a} + \mathbf{b}\Theta])^{1/2} \\ \tau[\] &\longrightarrow (\det[\mathbf{a} + \mathbf{b}\Theta])^{-1/2} \tau[\]. \end{aligned} \quad (4.30)$$

Automorphisms of the algebra \mathcal{A}_Θ require $\mathbf{b} = 0$, so that $\Theta = (\mathbf{c} + \mathbf{d}\Theta)\mathbf{d}^t$; they act on the basis elements $U_{\vec{n}}$ as $U_{\vec{n}} \rightarrow \chi(\vec{n})U_{\mathbf{d}^t \vec{n}}$, where $\chi(\vec{n}) \in U(1)$. It is known that Θ can be

skew-diagonalized over the integers. If the skew eigenvalues are not rationally related, and d is even, then the automorphism group is precisely $SL(2, \mathbb{Z})^{d/2} \rtimes \tilde{T}^d$.

The noncommutative torus algebra \mathcal{A}_Θ appears as a decoupled factor in the full string vertex algebra \mathcal{A}_{str} only in the zero slope limit $\alpha' \rightarrow 0$ [50]; away from this limit, one ought to consider \mathcal{A}_{str} itself and carry on the discussion at the level of string field theory. In the zero slope limit, there is still an action of the T-duality group on \mathcal{A}_Θ via (4.30). Furthermore, the stringy modes decouple from low-energy physics, leaving a noncommutative field theory built on \mathcal{A}_Θ . In terms of the Narain data, one is approaching the boundary of the moduli space where effectively $(G + \Phi) \rightarrow \infty$; the T-duality group acts (ergodically) via (4.30). So for orbifolds in which the volume remains as a modulus (symmetric orbifolds in particular), we can discuss the construction of D-branes as solitons consistently at the level of noncommutative field theory. Asymmetric orbifolds often fix the Narain data to some enhanced symmetry locus in the middle of moduli space, in which case this freedom is not available. For the rest of this subsection, we will restrict attention to orbifolds which allow a decoupling of the zero mode algebra to \mathcal{A}_Θ , and discuss the orbifold action there. The next subsection deals with asymmetric orbifolds.

Consider then symmetric orbifolds. The orbifold is again associated to a crossed product algebra $\mathcal{A}_\Theta \rtimes \mathcal{T}_{\text{fix}}$ [26]

$$\begin{aligned} U_{\vec{m}} U_{\vec{n}} &= e^{2\pi i \vec{m} \cdot \Theta \cdot \vec{n}} U_{\vec{n}} U_{\vec{m}} \\ \pi_G(g) U_{\vec{n}} \pi_G(g)^{-1} &= e^{i\chi(\vec{n}, g)} U_{R(g)\vec{n}} \\ \pi_G(g) \pi_G(h) &= \sigma(g, h) \pi_G(h) \pi_G(g) , \end{aligned} \tag{4.31}$$

where we have included the possibility of H -torsion in the form of cocycles χ, σ in the action of $G = \mathcal{T}_{\text{fix}}$; χ, σ can also arise when the orbifold group includes shifts on the noncommutative torus in addition to rotations, *c.f.* the discussion at the beginning of this section. Alternatively, one can take the orbifold of the noncommutative plane \mathbb{R}^d by the full space group of the orbifold. For symmetric orbifolds, the orbifold group G consists of crystallographic symmetries of the lattice defining the torus. These are the $SL(d)$ automorphisms defined above, together with shifts on the torus dual to the lattice. If we further demand invariance of the Hamiltonian, we are restricted to rotations $O(d) \subset SL(d)$.

The simplest examples to consider are orbifolds by symmetric shifts. From the discussion in section 4.1, the elements $U_{\vec{n}}$ are to be thought of as translations on the lattice of momenta $\mathbb{D} \cong \mathbb{Z}^d$ on the torus. One can think of the lattice \mathbb{D} as embedded in the

noncommutative plane \mathbb{R}^d parametrizing translations (2.3), and Θ as determined by the basis vectors of the lattice and the cocycle for translations in \mathbb{R}^d . A commutative torus would be defined by the dual lattice \mathbf{D}^* as $\mathbf{T}^d = \mathbb{R}^d / \mathbf{D}^*$; then an order N symmetric shift is a vector v such that $Nv \in \mathbf{D}^*$ is primitive. The orbifold by this symmetric shift decreases the volume of the torus by a factor N , and correspondingly increases the size of the lattice of momenta by eliminating all $p \in \mathbf{D}$ such that $p \cdot v \notin \mathbb{Z}$. It is this last fact that tells us how to implement the orbifold of the noncommutative torus by a symmetric shift. Recall that the physical string modes live in the commutant of G in the crossed product algebra; thus we simply need a group action that leaves invariant the same basis elements of \mathcal{A}_θ as in the commutative case. It is sufficient to take $G = \mathbb{Z}_N$, with generator g acting on $U_{\vec{n}}$ as

$$\pi_G(g)U_{\vec{n}}\pi_G(g)^{-1} = e^{2\pi i \vec{n} \cdot \vec{v}} U_{\vec{n}} \quad (4.32)$$

(a special case of (4.31)), where again $Nv \in \mathbf{D}^*$. The commutant of G in $\mathcal{A}_\Theta \rtimes G$ is the noncommutative algebra of the orbifolded torus.

Now we consider rotations. A major difference between the compact and noncompact orbifolds is the presence of multiple fixed points in the compact case. While the Hilbert space \mathcal{H} breaks up into isotypical components under the action of G , this does not reveal the entire fractional brane structure of the orbifold (roughly speaking, the isotypical components should only give the fractional brane structure of one of the fixed points). In addition to the obvious projectors

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g^{-1}) \pi_G(g) \quad (4.33)$$

onto isotypical components, there may be other embeddings $\varepsilon : G \rightarrow \mathcal{A}_\Theta \rtimes G$ of the orbifold group into the crossed product algebra. Let the image of $g \in G$ in the crossed product algebra be $\varepsilon(g)g$; then

$$\frac{1}{|G|} \sum_{g \in G} \chi_\rho(g^{-1}) \pi_{\mathcal{A}}(\varepsilon(g)) \pi_G(g) \quad (4.34)$$

is a distinct projector. The projectors (4.34) fall into a finite number of unitary equivalence classes (equivalent projectors reduce in the commutative case to fractional branes at covering space images of the same fixed point on the torus). Such projectors typically do not exhaust the set of fractional brane projectors of the torus orbifold, as there can be fixed loci consisting of orbits of points stabilized by some subgroup of G (see below).

Let us now turn to examples. Specialize again to the two-dimensional situation, and suppose that G does not contain shift vectors. Automorphisms of \mathcal{A}_θ are generated by $SL(2, \mathbb{Z})$ transformations mixing U and V : (U, V) and $(U^a V^b, U^c V^d)$ generate the same noncommutative torus algebra (4.1), provided $ad - bc = 1$. Rotations involved in symmetric orbifolds are in $O(2) \cap SL(2, \mathbb{Z})$

$$\begin{aligned}
\mathbb{Z}_2 : \quad & U \rightarrow U^{-1}, \quad V \rightarrow V^{-1} \\
\mathbb{Z}_3 : \quad & U \rightarrow V, \quad V \rightarrow U^{-1}V^{-1} \\
\mathbb{Z}_4 : \quad & U \rightarrow V, \quad V \rightarrow U^{-1} \\
\mathbb{Z}_6 : \quad & U \rightarrow V, \quad V \rightarrow U^{-1}V,
\end{aligned} \tag{4.35}$$

While *any* subgroup of $SL(2, \mathbb{Z})$ can serve as an automorphism group on \mathcal{A}_θ , we restrict attention to those groups which preserve the Hamiltonian. It turns out that these are the same as the finite subgroups of $SL(2, \mathbb{Z})$.

The general $SL(2, \mathbb{Z})$ transformation acts unitarily on \mathcal{H} . As in the noncompact case, the Hilbert space breaks up into $|G|$ isotypical components under the action of G , and one might imagine proceeding as in the previous section. Alternatively, a regular representation brane may be constructed by embedding a given projector, such as (4.11), in a $|G|$ -dimensional Chan-Paton space with successive entries rotated according to the G action (4.35) (*c.f.* (3.20)). For a different construction of modules of the crossed product in the \mathbb{Z}_2 case, see [26]. In the \mathbb{Z}_4 orbifold, in addition to the three nontrivial projectors (4.33) one has three more of the form (4.34) using $\varepsilon(g) = e^{i\pi\theta/2} U$, where g is the generator of \mathbb{Z}_4 [68]. In the commutative setting ($\theta = 0$), $\varepsilon(g)g$ represents a $\pi/2$ rotation followed by a lattice translation, so we might think of this second set of projectors as related to pointlike branes on the orbifold concentrated at the ‘point’ on the torus fixed by this transformation rather than at the ‘origin’ in the covering space fixed by g ; of course, this is rather imprecise language, since there are no ‘points’ in noncommutative geometry. In the \mathbb{Z}_4 example, the six projectors discussed above exhaust the set of independent projectors of the form (4.34) [68]. To complete the list of such projectors for two-dimensional toroidal orbifolds, for \mathbb{Z}_2 one has four projectors based on $\varepsilon(g) = 1, U, V, e^{i\pi\theta} UV$; for \mathbb{Z}_3 , one has six projectors based on $\varepsilon(g) = 1, U, V^{-1}$ (two each for two independent choices of a cube root of unity); and for \mathbb{Z}_6 there are only the five nontrivial projectors (4.33) onto the isotypical components of G .

Projectors of the type (4.34) characterize pointlike branes on the orbifold. In the \mathbb{Z}_2 case, one need only add the projector \mathbf{P}_θ (4.11) or (4.17) to span the lattice of K-theory

charges. In the \mathbb{Z}_4 case, one must use the \mathbb{Z}_4 invariant projector (4.17); three further projection operators, needed to span the lattice of K-theory charges, arise from (4.17) for different choices of the action of the generator g of \mathbb{Z}_4 on the module $\mathcal{S}(\mathbb{R})$ – $gf = \hat{f}$, if , or $-\hat{f}$ for $f \in \mathcal{S}(\mathbb{R})$, where \hat{f} is the Fourier transform of f . These describe two different branes associated to orbits of ‘points’ fixed by $\mathbb{Z}_2 \subset \mathbb{Z}_4$, but not by \mathbb{Z}_4 , as well as the ‘untwisted sector’ brane described by (4.17) before orbifolding. For \mathbb{Z}_3 , one needs only one nontrivial projector of the type (4.17), but invariant under the \mathbb{Z}_3 action (4.35); for \mathbb{Z}_6 , to completely span the lattice of K-theory charges requires four additional such projectors. We have not found explicit expressions for the projectors in these two cases.

4.5. Asymmetric orbifolds

A very interesting set of backgrounds are asymmetric toroidal orbifolds. These are obtained by gauging a subgroup \mathcal{T}_{fix} of the T-duality group $O(d, d)$ that leaves the Narain data fixed. In general this is only possible at enhanced symmetry points, which are in the middle of the Narain moduli space. Therefore, we can no longer simplify the analysis by taking the zero slope limit – the metric and B -field are necessarily of order the string scale. As a simple example, consider an orbifold by T -duality itself. In this case $E \rightarrow E^{-1}$ preserves the Narain data only when $B = 0$ and the size of the torus is string scale. Evidently, one can neither take the limit of [2] nor of [12]! A discussion of branes in asymmetric orbifolds along the above lines must therefore use the full noncommutative structure of the string field algebra. There is also typically no advantage to working with the noncommutative description in terms of G , Θ , and Φ . The noncommutativity inherent in the string scale will be of the same order as that due to Θ ; furthermore, the noncommutative data transform inhomogeneously (4.30). Transformations which fix a point $E = g + B$ in Narain moduli space generically will not fix the noncommutative description, thereby obscuring the presence of an enhanced symmetry. Therefore, we proceed formally with the standard open string vertex algebra in its usual description in terms of background data g , B , and make a proposal for how to formulate D-brane projectors.

We would like to build an algebra on which \mathcal{T}_{fix} acts as a group of automorphisms. A natural object in this regard is

$$\mathcal{B} = \begin{pmatrix} \mathcal{A}_{a_1 a_1} & \mathcal{M}_{a_1 a_2} & \cdots & \mathcal{M}_{a_1 a_n} \\ \mathcal{M}_{a_2 a_1} & \mathcal{A}_{a_2 a_2} & & \vdots \\ \vdots & & \ddots & \\ \mathcal{M}_{a_n a_1} & \cdots & & \mathcal{A}_{a_n a_n} \end{pmatrix}, \quad (4.36)$$

the *linking algebra* whose diagonal blocks are the open string vertex algebras $\mathcal{A}_{a_i a_i}^{\text{str}}$ for those boundary conditions a_i related by the action of the orbifold group \mathcal{T}_{fix} , and whose off-diagonal blocks are the Morita equivalence bimodules $\mathcal{M}_{a_i a_j}$ mapping $\mathcal{A}_{a_i a_i}^{\text{str}}$ to $\mathcal{A}_{a_j a_j}^{\text{str}}$ (*i.e.* the vertex operators creating strings that have boundary conditions a_i, a_j at either end). Note that we have a product $\mathcal{M}_{ab} \otimes_{\mathcal{A}_{bb}} \mathcal{M}_{bc} \rightarrow \mathcal{M}_{ac}$ so that (4.36) is indeed an algebra. A very similar construction is used in the theory of Morita equivalence of C^* algebras [69][61].

Naively one might expect that \mathcal{T}_{fix} embeds in the linking algebra (4.36) as a group of isomorphisms permuting the algebras $\mathcal{A}_{a_i a_i}$, however this is generically not the case. Morita equivalence is *not* an isomorphism of algebras, it is an equivalence relation, which is weaker. At generic points in Narain moduli space, the linking algebra has no automorphisms beyond those of its component algebras along the block diagonal. An instructive example is the action of the T-duality $E \rightarrow 1/E$. The open string algebras depend parametrically on the Narain data E as well as the boundary conditions a_i ; T-duality acts for example to relate the open string algebra with Neumann boundary conditions in background E to the open string algebra with Dirichlet boundary conditions in background $1/E$: $\mathcal{A}_{NN}^{\text{str}}(E) \cong \mathcal{A}_{DD}^{\text{str}}(1/E)$. This isomorphism relates string algebras with different boundary conditions at different points in Narain moduli space. It is not an automorphism of the linking algebra, which packages together open string algebras for different boundary conditions at fixed E , except at the fixed point $E = 1/E$. At such special points T-duality acts as an automorphism of the linking algebra $\mathcal{B}(E_{\text{fix}})$.

Thus the linking algebra carries a natural action of \mathcal{T}_{fix} , as an extra group of automorphisms, precisely at the enhanced symmetry points of Narain moduli space. This leads us to conjecture that the desired orbifold algebra is a direct summand of $(\mathcal{B}(E_{\text{fix}}) \rtimes \mathcal{T}_{\text{fix}})^{\mathcal{T}_{\text{fix}}}$. The regular representations given by the collections of vertex operators with boundary conditions $a \in \{a_i\}$ split into isotypical components according to the irreducible representations of \mathcal{T}_{fix} . Presumably, there are other fractional brane projectors, for instance of the type (4.34); the projection operators should span the lattice of K-theory charges. Note that resolving the representations of (4.36) into isotypical components justifies the formal sums of Dp -branes of different p required to make fractional branes (particular examples have been studied in [70]).

We leave the nontrivial problem of the classification of projectors and the formulation of the associated K-theory of general asymmetric orbifolds to future work. There is, however, a simple example where we can carry out the discussion at the level of the zero

mode algebra (in its noncommutative description as \mathcal{A}_Θ) in the decoupling limit – namely, orbifolds by asymmetric shifts. In order to formulate it, let us consider first the following construction. Consider a noncommutative torus algebra $\tilde{\mathcal{A}}_{\tilde{\Theta}}$ graded by a lattice $\tilde{\mathbf{D}} \subset \mathbb{R}^d$. Let $\mathbf{D} \subset \tilde{\mathbf{D}}$ be a sublattice with $\tilde{\mathbf{D}}/\mathbf{D} \cong \mathbb{Z}_N$, so that

$$\tilde{\mathbf{D}} = \mathbf{D} \oplus (\mathbf{D} + w) \oplus (\mathbf{D} + 2w) \oplus \cdots \oplus (\mathbf{D} + (N-1)w) \quad (4.37)$$

where $w \in \tilde{\mathbf{D}}$ with $Nw \in \mathbf{D}$. Let \mathcal{A}_Θ be the subalgebra generated by U_p , with $p \in \mathbf{D}$. Note that if \tilde{e}_i is a basis for $\tilde{\mathbf{D}}$ then a basis e_i for \mathbf{D} will be given by $e_i = S_i^j \tilde{e}_j$ where S is a matrix of integers with nonzero determinant. Then $\Theta = S^t \tilde{\Theta} S$. How can we construct $\tilde{\mathcal{A}}_{\tilde{\Theta}}$ from \mathcal{A}_Θ ? We claim that $\tilde{\mathcal{A}}_{\tilde{\Theta}}$ is the \mathbb{Z}_N invariant subalgebra of a linking algebra for \mathcal{A}_Θ . To see this note that since U_w normalizes \mathcal{A}_Θ

$$\mathcal{M}_{mn} := U_{mw} \mathcal{A}_\Theta U_{-nw} \quad (4.38)$$

is a bimodule for \mathcal{A}_Θ , where $m, n \in \mathbb{Z}_N$. Indeed, we could also write it as $\mathcal{M}_{mn} \cong \mathcal{M}_{(m-n) \bmod N}$ where

$$\mathcal{M}_k := \text{Span}\{U_p | p \in kw + \mathbf{D}\} \quad (4.39)$$

Moreover, as is evident from (4.38), there is a multiplication $\mathcal{M}_{mn} \otimes \mathcal{M}_{nm'} \rightarrow \mathcal{M}_{mm'}$. Therefore, we may use the bimodules (4.38) to form the linking algebra (4.36) above. Now, the group \mathbb{Z}_N acts as a group of automorphisms on the linking algebra by taking $B \rightarrow \mathcal{G} B \mathcal{G}^{-1}$ where \mathcal{G} is the element

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \gamma \\ \gamma & 0 & & & \vdots \\ 0 & \gamma & \ddots & & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & \gamma & 0 \end{pmatrix}, \quad \gamma = U_w. \quad (4.40)$$

Now we have

$$\tilde{\mathcal{A}}_{\tilde{\Theta}} = \mathcal{B}^{\mathbb{Z}_N}. \quad (4.41)$$

The lattice of momenta $\tilde{\mathbf{D}}$ is precisely what results from the orbifold by an asymmetric shift by a fractional winding w , such that Nw is an allowed closed string winding vector in the Narain lattice (the latter is isomorphic to the unique self-dual Lorentzian lattice $II^{d,d}$). Of course, in this example we could have used the Morita equivalence to the T-dual algebra $\mathcal{A}_{1/\Theta}$, and performed a symmetric shift (4.32) there; however, in the general orbifold

group including both asymmetric twists and shifts, we will need to consider algebras (4.36) containing T-dual algebras and this procedure will not be available.

It would be interesting to consider the most general asymmetric shift v , $Nv \in II^{d,d}$; however, this would take us into a lengthy detour into the specifics of shift vectors, level matching constraints, and the like, and so we will not pursue it here. A few useful remarks about orbifolds by shifts are collected in Appendix B.

5. Conclusion: Some future directions

D-branes are sources of RR fields and hence carry RR charge. The RR charges are neatly summarized by the Chern-Simons coupling in the D-brane worldvolume Lagrangian. It is natural to ask whether such couplings can be formulated in the context of noncommutative geometry. We plan to address this question in [30].

In this paper we have examined rather simple orbifolds and crossed-product algebras. Given the examples which tend to be studied in the C^* -algebra literature it is natural to ask whether orbifolds by other infinite discrete groups (for example, non-amenable groups) or by ergodic actions of real Lie groups might provide interesting examples of string backgrounds. A key requirement in formulating orbifolds in string theory is that the orbifold group must be a symmetry of the dynamics, so that we can gauge it. For this reason, it is unlikely that one can make a sensible string background based on foliations of tori. Nevertheless, there are some backgrounds and limits of string theory where the Hamiltonian is effectively zero and where one might consider more nontrivial crossed-product algebras. One particularly interesting example might be formulating string theory on quotients of $T^*SL(2, \mathcal{R})$ by infinite discrete groups of hyperbolic isometries. Similar quotients were considered as cosmological models in [71]. If such backgrounds make sense then the viewpoint of this paper should prove useful for the formulation of the corresponding D-branes. Such orbifolds will not preserve any supersymmetry, and generically will be expected to have tachyons; nevertheless, the classical string theory is well-defined and might be of some interest.

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Appendix A. Morita equivalence bimodules on higher-dimensional tori

Here we summarize results of Rieffel [60] on the construction of modules for the noncommutative torus algebra \mathcal{A}_Θ (4.7) on tori of arbitrary dimension. Consider somewhat more generally any locally compact abelian group M , its dual group \hat{M} , and let $G = M \times \hat{M}$. On G there is a canonical cocycle

$$\beta((s, t), (v, w)) = \exp[2\pi i \langle s, w \rangle] \quad , \quad s, v \in M \quad , \quad t, w \in \hat{M} \quad , \quad (\text{A.1})$$

and a projective representation of G on the space of Schwarz functions $\mathcal{S}(M)$ via

$$U_{(s,t)} f(v) = e^{2\pi i \langle v, t \rangle} f(v + s) \quad . \quad (\text{A.2})$$

From this one easily deduces

$$\begin{aligned} U_x U_y &= \beta(x, y) U_{x+y} \\ U_x U_y &= \beta(x, y) \bar{\beta}(y, x) U_y U_x \equiv \rho(x, y) U_y U_x \end{aligned} \quad (\text{A.3})$$

for $x = (s, t), y = (v, w) \in G$.

Now take $M = \mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{Z}_r$ (so $\hat{M} = \hat{\mathbb{R}}^p \times \mathbf{T}^q \times \hat{\mathbb{Z}}_r$), and $2p + q = d$; and represent it as in (A.2) on $\mathcal{S}(M)$. Suppose $D \subset G$ is a cocompact group, with $D \cong \mathbb{Z}^d$, and such that the elements $U_{(s,t)}$ with $(s, t) \in D$ generate \mathcal{A}_Θ . Then, since D is cocompact, the orthogonal complement determined by

$$D^\perp = \{y \in G : \rho(x, y) = 1 \quad \forall x \in D\} \quad (\text{A.4})$$

defines an algebra $\mathcal{B} = \mathcal{A}_{\Theta'}$ Morita equivalent to $\mathcal{A} = \mathcal{A}_\Theta$. In the 2d example of section 4, we had $p = 1, q = r = 0$; thus $G = \mathbb{R} \times \hat{\mathbb{R}}$, with cocycle

$$\rho_{2d}((s, t), (v, w)) = \exp[2\pi i (sw - tv)] \quad ; \quad (\text{A.5})$$

the lattices D and D^\perp are $D = \{(s, t) = (j\theta, k), j, k \in \mathbb{Z}\}$ and $D^\perp = \{(v, w) = (m, n/\theta), m, n \in \mathbb{Z}\}$. Indeed the algebra $\mathcal{A}_{\theta'}$ defined by D^\perp is Morita equivalent to \mathcal{A}_θ by $\theta \rightarrow -1/\theta$.¹⁴ The construction agrees with the analysis of [2] which showed that the zero modes of 2-2 strings live in a Hilbert space of functions on \mathbf{T}^2 (the Fourier dual

¹⁴ The representation of the algebra (4.15) is the conjugate or opposite algebra of (A.2) because it acts oppositely (on the left vs. the right).

to \mathbb{Z}^2 used above for $p = 0, q = 2$), whereas the zero modes of 2-0 strings live in a Hilbert space of functions on \mathbb{R} . Physically, the NN coordinates describe quantized momenta, the DD coordinates have quantized winding, and hence their zero mode momenta or winding lives on the appropriate lattice. On the other hand, the DN strings' zero modes describe a rigid rotator tethered on one end, whose free end lives on the covering space of the torus. The wavefunction for the free end will thus depend on only half of the coordinates. Thus the condition $2p + q = d$.

Functions $f, g \in \mathcal{S}(M)$ yield elements of the algebra $\mathcal{A} = \mathcal{A}_\Theta$ just as in the two-dimensional case (4.14) via

$$\langle f, g \rangle_{\mathcal{A}} = \sum_{(s,t) \in \mathcal{D}} \left(\int_M dv \overline{f(v+s)} g(v) e^{2\pi i \langle v, t \rangle} \right) U_{(s,t)}, \quad (\text{A.6})$$

and similarly for $\mathcal{B} = \mathcal{A}_{\Theta'}$; the Morita equivalence relation

$$\langle f, g \rangle_{\mathcal{B}} h = f \langle g, h \rangle_{\mathcal{A}} \quad (\text{A.7})$$

for $f, g, h \in \mathcal{S}(M)$, follows by a Poisson summation formula relating sums over \mathcal{D} and \mathcal{D}^\perp [60].

Now one may choose a *finite set* [60] of elements $f_1, \dots, f_n \in \mathcal{S}(M)$ such that

$$\sum_i \langle f_i, f_i \rangle_{\mathcal{B}} = \mathbf{1}_{\mathcal{B}}; \quad (\text{A.8})$$

then the \mathcal{A} -valued inner products

$$\mathbf{P}_{ij} = \langle f_i, f_j \rangle_{\mathcal{A}} \quad (\text{A.9})$$

are the matrix elements of a projector $\mathbf{P} \in \text{Mat}_n(\mathcal{A})$, and [60] shows that all \mathcal{A} modules are obtained in this way. In this construction, one finds that $2p$ is the rank of the highest nonzero Chern class of the module; the role of the finite group $\mathbb{Z}_r \times \hat{\mathbb{Z}}_r \subset G$ is to allow for twisted boundary conditions, as in the discussion of boundary conditions for 2- (n,m) strings in section 6.3 of [2] (see also [31,5,6,9,26]).

Appendix B. Comments on shifts

One way to write the Narain lattice is as follows. Take the standard Euclidean metric on $\mathbb{R}^d \oplus \mathbb{R}^d$ of signature $(1, -1)$. Then

$$\begin{aligned} e^a &= \frac{1}{\sqrt{2}}(e^a{}_\nu ; e^a{}_\nu) \\ f_a &= \frac{1}{\sqrt{2}}(e_a{}^\nu E_{\nu\mu} ; -e_a{}^\nu E_{\nu\mu}^t) \end{aligned} \tag{B.1}$$

with $a, \mu = 1, \dots, d$, span a lattice isomorphic to $II^{d,d}$. (Here take $g_{\mu\nu} = \delta_{\mu\nu}$, and $E = g + B$.) Put differently,

$$\mathbf{E} := \frac{1}{\sqrt{2}} \begin{pmatrix} e^a{}_\nu & e^a{}_\nu \\ e_a{}^\nu E_{\nu\mu} & e_a{}^\nu E_{\nu\mu}^t \end{pmatrix} \tag{B.2}$$

satisfies

$$\mathbf{E} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{E}^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.3}$$

The lattice metric is $g^{ab} = e^a{}_\mu e^b{}^\mu$, where $e_a{}^\mu e^b{}_\mu = \delta_a{}^b$.

Let $p = m_a e^a + w^a f_a$, $m_a, w^a \in \mathbb{Z}$ be the generic lattice vector in the lattice spanned by e^a, f_a . Suppose we have a shift vector so that $Nv \in II^{d,d}$ is primitive. Then, provided the level matching condition $Nv^2 \in 2\mathbb{Z}$ is satisfied, the lattice defined by taking the projection $p \cdot v \in \mathbb{Z}$ together with the union of cosets $(p + \ell v)$ (projected to the vectors with $(p + \ell v)^2 \in 2\mathbb{Z}$) defines another even unimodular lattice. This lattice will be spanned by \bar{e}_a, \bar{f}^a related to the original basis vectors by

$$\begin{pmatrix} \bar{e}^a \\ \bar{f}_a \end{pmatrix} = S \begin{pmatrix} e^a \\ f_a \end{pmatrix} \tag{B.4}$$

where S is a rational matrix such that

$$S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.5}$$

One way to prove this is to use the Narain-Siegel theta function. The shift has thus produced an orbifold at another point in Narain moduli space specified by $\mathbf{E}' = S\mathbf{E}$.

To make the above construction more explicit we need to say what S is. We can be slightly more explicit if we consider an orbifold with a shift of the form $v = \frac{1}{N}v^a f_a$, with $v^a \in \mathbb{Z}$. The condition $p \cdot v = 0 \bmod 1$, becomes $m_a v^a = 0 \bmod N$. This condition defines an integral sublattice of the d -dimensional lattice $m_a e^a$. This new lattice will be spanned

by vectors $\bar{e}^a = s^a_b e^b$ where the matrix s is a matrix of integers. Let the d -dimensional lattice spanned by the \bar{e}^a be denoted by L .

Now let F be the lattice spanned by f_a . Then $F, F+v, F+2v, \dots$ spans a Euclidean lattice. We claim this is the dual lattice L^* . One way to prove this is the following. For the purposes of this argument we introduce an auxiliary metric on the span of f_a such that $\langle\langle f_a, f_b \rangle\rangle = \delta_{ab}$ and form the theta function of the lattice. Similarly we introduce such an auxiliary metric on L , that is on the span of e^a , $\langle\langle e^a, e^b \rangle\rangle = \delta^{ab}$. Then from the definition of L we have

$$\Theta_L = \sum_{m_a \in \mathbb{Z}} e^{i\pi\tau m_a^2} \left(\frac{1}{N} \sum_{\ell=0}^{N-1} e^{2\pi i m_a v^a \ell / N} \right) \quad (\text{B.6})$$

On the one hand, $\Theta_L(-1/\tau) = \sqrt{|L/L^*|} \Theta_{L^*}(\tau)$; on the other hand, doing the modular transformation term-by-term gives the description in terms of F and its translates by v . The lattice $F \oplus (F+v) \oplus (F+2v) \oplus \dots$ has a basis $\bar{f}_a = (s^{tr, -1})^b_a f_b$.

Thus, the lattice after the shift is spanned by vectors \bar{e}^a, \bar{f}_a which are related to the original vectors e^a, f_a by

$$\begin{pmatrix} \bar{e}^a \\ \bar{f}_a \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & (s^t)^{-1} \end{pmatrix} \begin{pmatrix} e^a \\ f_a \end{pmatrix} \quad (\text{B.7})$$

Note that the matrix is a rational matrix in $O(d, d)$. Now, the new point in Narain moduli space is

$$\mathbf{E}' = \begin{pmatrix} s & 0 \\ 0 & (s^t)^{-1} \end{pmatrix} \mathbf{E} \quad (\text{B.8})$$

This is just an $GL(d, \mathbb{Q})$ transformation of the torus, holding $B_{\mu\nu}$ in the covering space coordinates fixed. Similar remarks apply to shift vectors which are of the form $v = \frac{1}{N} v_a e^a$.

For example, consider $v = \frac{1}{N}(ae+bf)$ in the 1d case. Let us suppose that ab is nonzero. By level matching ab divides N . Allowed momenta in the untwisted sector satisfy $p \cdot v \in \mathbb{Z}$, for $p = ke + wf$, $k, w \in \mathbb{Z}$; in other words, $k \in \frac{N}{(N, b)} \mathbb{Z}$, $w \in \frac{N}{(N, a)} \mathbb{Z}$. The twisted sectors add all $k \in \frac{(N, a)}{(N, b)} \mathbb{Z}$ and $w \in \frac{(N, b)}{(N, a)} \mathbb{Z}$, so that the net effect is that the radius of the circle has been rescaled by $s = \frac{(N, b)}{(N, a)}$.

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